① Ensembles, Estimators, Gaussians HW 2 Answers ($\boxed{11}$ - $\boxed{15}$)

1. From pp 16 - 18: $C_1(x) = 1$.

$$p(x) = B e^{-\lambda x}.$$

Need $\int_a^b p(x)dx = 1$, $\int_a^b x p(x) dx = M$.
(1) (2)

$$(1): \quad -\frac{B}{\lambda} e^{-\lambda x}\Big|_a^b = 1 \Rightarrow \frac{1}{B} = \frac{1}{\lambda}\left(e^{-\lambda a} - e^{-\lambda b}\right)$$

$$(2): \quad -\frac{Bx}{\lambda} e^{-\lambda x} + \frac{B}{\lambda^2} e^{-\lambda x}\Big|_a^b = M$$

$$B\left(\frac{1}{\lambda}\left(a e^{-\lambda a} - b e^{-\lambda b}\right) - \frac{1}{\lambda^2}\left(e^{-\lambda a} - e^{-\lambda b}\right)\right) = M$$

$$\frac{a e^{-\lambda a} - b e^{-\lambda b}}{e^{-\lambda a} - e^{-\lambda b}} - \frac{1}{\lambda} = M$$

$\lambda \to +\infty$: LHS $\to a$.
$\lambda \to -\infty$: LHS $\to b$.
LHS is continuous, so there must be some λ that solves.

② Ensembles, Estimates, Gaussians HW 2 Answer (III - IV)

2 For a Gaussian of variance σ^2, & mean 0,

namely, $$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}$$

the 4^{th} moment is given by

$$\int_{-\infty}^{\infty} x^4 p(x)\,dx = 3\sigma^4!$$

[Detour: moments of a central Gaussian:

$$M_n = \int_{-\infty}^{\infty} x^n p(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \underbrace{x^{n-1}}_{u} \; \underbrace{x\, e^{-x^2/2\sigma^2}\, dx}_{dv}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \Big[x^{n-1} \cdot \left(-e^{-x^2/2\sigma^2} \cdot \sigma^2\right)$$

$$+ \sigma^2 \int_{-\infty}^{\infty} (n-1)\, x^{n-2} e^{-x^2/2\sigma^2}\, dx \Big]$$

$$= \sigma^2 M_{n-2} \cdot (n-1)$$

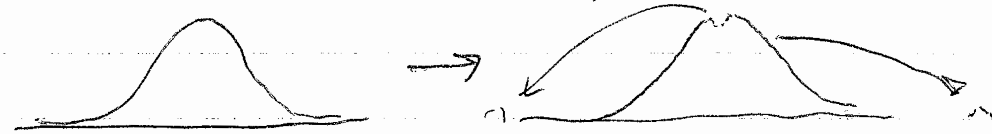
So $M_0 = 1$, $M_2 = \sigma^2$, $M_4 = 3\sigma^4$, $M_6 = 15\sigma^6$, ...]

 Ensemble, Estimating Gaussians HW2 Answers ([illegible] – [illegible])

2, ctd.

By constraining the 4th moment, we can only make a distribution that is less entropic than a Gaussian.

If 4th moment $> 3\sigma^4$, can move a little bit of mass further away from 0:



By moving "little bits" further and further away, we can make a distrib. arbitrarily close to a Gaussian in entropy, but with a 4th moment $> 3\sigma^4$!

[two degrees of freedom; how much mass, how far to move]

So, for 4th moment $> 3\sigma^4$, no maxent distribution.

For 4th moment $< 3\sigma^4$, there will be a maxent distribution, of form

$$B\, e^{-(\alpha_1 x^2 + \alpha_2 x^4)}.$$

This will have thinner tails than a Gaussian ($\alpha_2 > 0$).

Similar situation for 3rd moment.