③ Fourier Analysis - Applications, ctd.

Another view of the fact that a linear system maps sinusoids to sinusoids (possibly of different amplitudes and phases):

Consider the differential equation

$$(*) \qquad \frac{d^2 y}{dt^2} + \omega^2 y = 0.$$

Solutions are the sinusoids $A e^{i(\omega t + \varphi)}$

Can write $(*)$ as $(D^2 + \omega^2 I) y = 0$,
where
$D \equiv \frac{d}{dt}$.

Observe that any linear time-invariant operator L commutes with D, since

$$(Dy)(t) = \lim_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{D_{\Delta t}(y) - y}{\Delta t}, \text{ i.e.}$$

$$D = \lim_{\Delta t \to 0} \frac{1}{\Delta t}(D_{\Delta t} - I)$$

L commutes $\bar{c}$ $D_{\Delta t}$ and I, so $\therefore$ it commutes $\bar{c}$ D.

Since L commutes $\bar{c}$ D, it commutes $\bar{c}$ $D^2 + \omega^2 I$

$$(D^2 + \omega^2 I) y = 0 \Rightarrow L(D^2 + \omega^2 I) y = 0$$

$$\Rightarrow (D^2 + \omega^2 I) L y = 0 \Rightarrow Ly \text{ is a sinusoid}$$
of freq ω.

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This argument generalizes:

for any differential eq'n with constant coefficients,

if

$$\left(\sum_0^N c_n D^n\right) y = 0, \text{ then } \left(\sum_0^N c_n D^n\right) L y = 0.$$

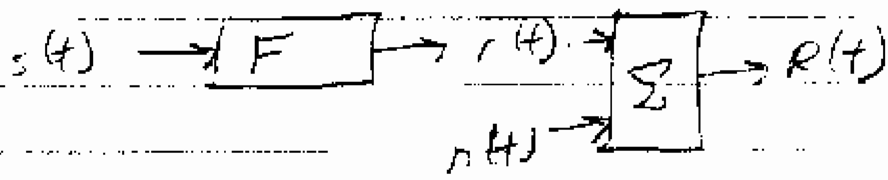
That is, why not use the solutions of $\left(\sum_0^N c_n D^n\right) y = 0$ as the analysis signals and fundamentals?

Solutions of $\left(\sum_0^N c_n D^n\right) y = 0$ are of the form

$$y = \sum_1^N a_r e^{\lambda_r t}, \text{ where } \lambda_r \text{ is a root of } \sum c_n \lambda^n = 0, \text{ and } a_r \in \mathbb{C}.$$

So - While this generalization is formally valid, the problem is that if $\operatorname{Re} \lambda_r \neq 0$, then either y goes to ∞ at + times, or at − times.

(15) How to measure $F(\omega)$?



$n(t)$ = "noise"; we have access to $R(t) = r(t) + n(t)$

$s(t)$ strictly periodic with period $\frac{2\pi}{\omega_0}$ $(s(t) = e^{i\omega_0 t})$
$r(t)$ " " " " "

But $n(t)$ is not periodic, nor is $R(t)$

So we can't quite use the formalism (b)
[periodic + continuous time] as applied to $R(t)$,
but,
writing $\langle R(t) \rangle$ = "periodic average" =

$$\lim_{N \to \infty} \frac{1}{N} \sum_{m=0}^{N-1} R(t + mP) \quad \left[P = \frac{2\pi}{\omega_0}\right]$$

we have

$$\langle R(t) \rangle = \langle r(t) \rangle + \langle n(t) \rangle$$

Provided $\langle n(t) \rangle = 0$

$$\langle R(t) \rangle = r(t)$$

Don't need to assume anything about higher moments of $n(t)$, e.g., $\langle n(t) n(t+\tau) \rangle_t$

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Then

(b): $\hat{x}_k = \int_0^1 e^{-2\pi i k \tau} x(\tau) d\tau$

with $\tau = \frac{\omega_0 t}{2\pi}$, $x(\tau) = s\left(\frac{2\pi \tau}{\omega_0}\right) = e^{i\omega_0 t}$

$$\hat{s}_k = \frac{1}{P}\int_0^P e^{-ikt\omega_0} s(t)\,dt$$

$\hat{s}_1 = 1$, other $\hat{x}_k' = 0$

Since $\hat{r} = \hat{F}\hat{s}$, $\hat{r}_1 = \hat{F}_1 = \hat{F}(\omega_0)$

But also $\hat{r}_1 = \frac{1}{P}\int_0^P e^{-it\omega_0} r(t)\,dt$

$$\hat{F}(\omega_0) = \frac{1}{P}\int_0^P e^{-it\omega_0} \langle R(t) \rangle dt$$

Could also have used formalism (c) continuous time from $-\infty$ to ∞ & written

$$\hat{s}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} e^{i\omega_0 t} dt = 2\pi\delta(\omega-\omega_0)$$

and measured $R(t)$ over "$[-\infty, \infty]$",

$$\hat{R}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} R(t) dt = \hat{F}(\omega_0)\cdot 2\pi\delta(\omega-\omega_0)$$

Avoid the δ-function $\Longleftrightarrow$ replace $\int_{-\infty}^{\infty}$ by

$$\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2}$$

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Comments on $\hat{F}(\omega_0) = \frac{1}{P}\int_0^P e^{-it\omega_0} \langle R(t)\rangle dt$

- This a "plug-in" estimator that is unbiased -- fundamentally, because it is linear in the data, and $\langle n(t)\rangle = 0$.

- We could use $\frac{1}{P}\int_0^P e^{-it\omega_0 k} \langle R(t)\rangle dt$

  as an estimate of nonlinearity ($k \neq 0$).

- We could use $s(t) = \sum_{k=1}^{Q} \alpha_k e^{i\omega_k t}$ and estimate

  $$\hat{F}(\omega_k) = \frac{1}{P_k}\cdot\frac{1}{\alpha_k}\int_0^{P_k} e^{-it\omega_k} \langle R(t)\rangle_k dt$$

  where $\langle R(t)\rangle_k$ is period-averaged with respect to $\frac{2\pi}{\omega_k}$.

Here $s(t)$ is not strictly periodic, the response to the frequencies ω_l ($l \neq k$) can be viewed as "noise" that does not corrupt the estimate of $\hat{F}(\omega_k)$.

In Fourier domain, $\hat{S}(\omega) = 2\pi \sum_{k=1}^{Q} \alpha_k \delta(\omega - \omega_k)$

$$\hat{r}(\omega) = 2\pi \sum_{k=1}^{Q} \alpha_k \hat{F}(\omega_k)\delta(\omega - \omega_k)$$

This measures multiple $F(\omega_k)$'s simultaneously

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- Consider a "spiking neuron"

$$s(t) \longrightarrow \boxed{L} \longrightarrow r(t) \longrightarrow \text{spike generation} \longrightarrow R(t)$$

Consider
$R(t) = \sum \delta(t - t_i)$, but the t_i's depend on the "instance"!

Provided
$\langle R(t) \rangle = r(t)$, above procedure works.

So, if $r(t)$ = "instantaneous rate"

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \langle \text{prob of a spike in } [t, t+\Delta t] \rangle$$

then above works.
[Cannot infer characteristics of spike generation from $\hat{F}$]

- Above analysis says nothing about the scatter of the estimate of $\hat{F}$.

This will depend on the statistics of $n(t)$,
[or the characteristics of spike generation]

The transfer function as a complex-analytic entity

$$\hat{F}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} F(t)\,dt = \int_{0}^{\infty} e^{-i\omega t} F(t)\,dt$$

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\,\hat{F}(\omega)\,d\omega$$

$$F(t) \text{ real} \iff \hat{F}(-\omega) = \overline{\hat{F}(\omega)}$$

But what are the implications of $F(t) = 0$ for $t < 0$?

$\boxed{\text{Cauchy's Thm}}$ If $g(z)$ is "analytic," then

$$g(a) = \frac{1}{2\pi i}\oint \frac{g(z)}{z-a}\,dz$$

Contour runs ctr-clockwise, & encloses a.

A complex-valued fn $f(z)$ is "analytic" in $\mathbb{C}$ if it is equal to its Taylor expansion anywhere, i.e.,

$$f(z) = \sum_{n=0}^{\infty} \frac{c_n}{n!}(z-z_0)^n, \quad c_n = \left.\frac{d^n f}{dz^n}\right|_{z=z_0}$$

[sometimes, say "$f(z)$ analytic in S", for S some subset of $\mathbb{C}$]

$f(z)=\sum_0^M a_n z^n$ is analytic (T.S. is finite)

$f(z)=e^z$, $\cos z$, $\sin z$ analytic

$f(z)=|z|$ not analytic at $z=0$ (no 1st deriv.)
$f(z)=z^{1/2}$ " " " " "
$f(z)=\ln z$ " " " " "

$f(z)=e^{\frac{1}{z^2}}$ " " " ", even though on real axis, all derivs exist - since all deriv's approach 0 as $z\to 0$.

Plausibility argument for Cauchy's Thm:

(a) Say $f(z)$ is analytic. Then $\oint f(z)\,dz=0$

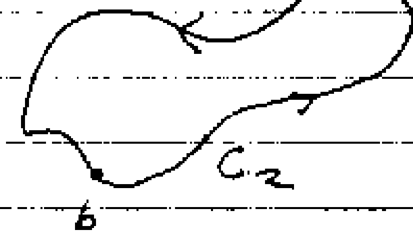
provided that the contour ○ stays within the region of analyticity.

Formally, if $f(z)=\sum_0^\infty \frac{c_n}{n!} z^n$, then

$\frac{dF}{dz}=f(z)$, for $F(z)=\sum_0^\infty \frac{c_n}{(n+1)!} z^{n+1}$.

$$\oint f(z)\,dz=\int_{a \text{ on } C_1}^{b} f(z)\,dz+\int_{b \text{ on } C_2}^{a} f(z)\,dz$$

$$=F(z)\Big|_a^b+F(z)\Big|_b^a=0$$



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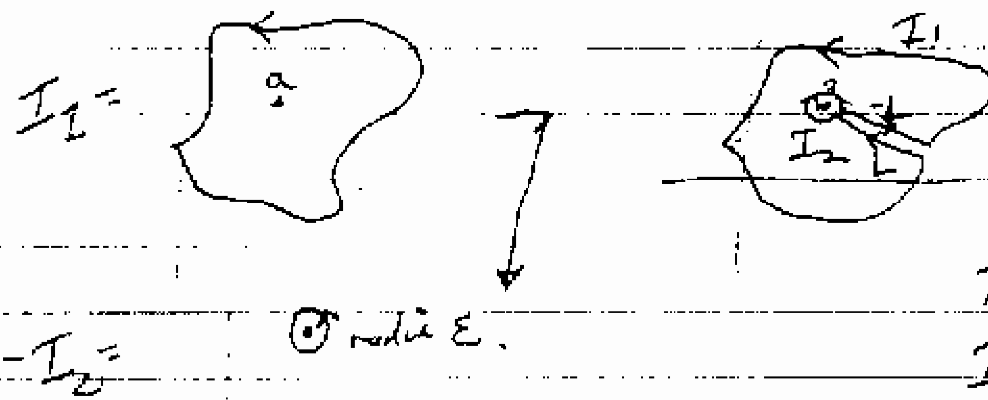
We just showed that the expression $\int_a^b f(z)\,dz$ is independent of path, provided f is analytic.

2.) If $g(z)$ is analytic, then so is $\frac{g(z)}{z-a}$ except at a.
[Show $\frac{1}{z-a}$ is analytic except at $z=a$, & show that the product of analytic fns is analytic]

3.) Shrink the arbitrary contour in Cauchy's Thm to one of arbitrarily small radius ε around a:



Path integral must be 0. This is $I_1 + I_2 + L - L$. So $I_1 = -I_2$

4.) $g(z)$ analytic at $z=a \Rightarrow g(z) = \sum_{n=0}^{\infty} \frac{c_n}{n!}(z-a)^n$; $c_0 = f(a)$

$$\frac{1}{2\pi i}\int_{\varepsilon\text{-circle}} \frac{g(z)}{z-a}\,dz = \frac{1}{2\pi i}\int_{\varepsilon\text{-circle}} \sum_{n=0}^{\infty} \frac{c_n}{n!}(z-a)^{n-1}\,dz$$

$$= \frac{1}{2\pi i}\int_{\varepsilon\text{-circle}} \frac{f(a)\,dz}{z-a} + \int_{\varepsilon\text{-circle}} \sum_{n=1}^{\infty} \frac{c_n}{n!}(z-a)^{n-1}\,dz$$

Second term is the integral of an analytic function around a closed contour, $\therefore$ must be 0.

Deal with first term, by taking $z = a + \varepsilon e^{i\theta}$, $\theta: 0$ to 2π

$$\frac{1}{2\pi i}\int_{\varepsilon\text{-circle}} \frac{g(z)}{z-a}\,dz = \frac{f(a)}{2\pi i}\int_{\varepsilon} \frac{dz}{z-a} = \frac{f(a)}{2\pi i}\int_0^{2\pi} \frac{\varepsilon i e^{i\theta} d\theta}{\varepsilon e^{i\theta}}$$

$$dz = \varepsilon i e^{i\theta} d\theta$$

$$= \frac{f(a)}{2\pi i}\int_0^{2\pi} i\,d\theta = f(a).$$

Note that $\displaystyle\frac{1}{2\pi i}\oint \frac{g(z)}{(z-a)^k}\,dz = 0$ for $k > 1$,

since at this final stage we'd have

$$\frac{f(a)}{2\pi i}\int_0^{2\pi} \frac{\varepsilon i e^{i\theta} d\theta}{(\varepsilon e^{i\theta})^k} = \frac{f(a)}{2\pi}\int_0 \varepsilon^{k-1} e^{i\theta(k-1)}\, d\theta.$$

Consequence of Cauchy's Thm: Residues

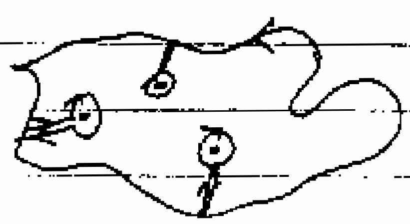
f: analytic except at isolated points $z_1, \dots, z_R$

At each z_k, assume $f(z) = \sum_{-n_k}^{\infty} a_n (z-z_k)^n$; $a_{-1} \equiv \operatorname{Res}_{z_k}(f)$.

("poles")

Then $\displaystyle\oint f(z)\,dz = 2\pi i \sum_{k=1}^{R} \operatorname{Res}_{z_k}(f).$

"Residue Thm"



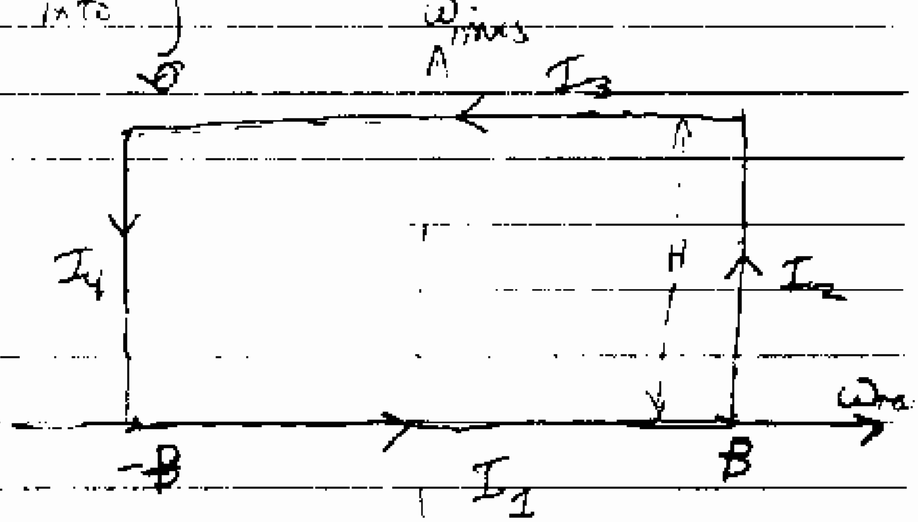
We want to apply this to $F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\hat{F}(\omega)\,d\omega$.

Is $\hat{F}(\omega)$ analytic? Since $\hat{F}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} F(t)\,dt$,
it is a continuous superposition
of analytic fns $e^{-i\omega t}$, one for each t.
Discrete superposition $\Rightarrow$ guaranteed analytic
Continuous superposition $\Rightarrow$ analytic only if $F(t)$ is
not bizarre - no discontinuities worse than
δ-functions AND moments $< \infty$.
Moments $< \infty$ guaranteed if $|F(t)| < Ke^{-\alpha t}$ for
some α (weaker conditions suffice)

So -- $F(t)$ has "exponential forgetting" $\Rightarrow$ $\hat{F}(\omega)$ analytic
$\hat{F}(\omega)$ analytic $\Rightarrow$ $e^{i\omega t}\hat{F}(\omega)$ analytic.

Next: turn $\int_{-\infty}^{\infty}$ into $\oint$

A. $t \geq 0$.



Want to show that I_2, I_3, I_4 can be made arbitrarily
small for sufficiently large B, H

Consider (on I_3) $e^{i\omega t}\hat{F}(\omega)$. $\omega = x + iH$.

$e^{i\omega t} = e^{i(x+iH)t} = e^{-tH} e^{ixt}$

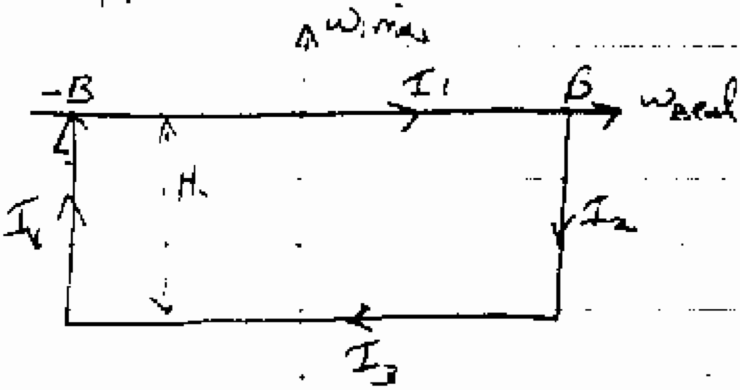
So, as $H \to \infty$, this $\to 0$.

Similar argument on I_2, I_4.

Since $I_2, I_3, I_4 \to 0$, the residue theorem applies.

$$* \quad \boxed{F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\hat{F}(\omega)\,d\omega = i\sum_{\text{at Poles in UHP}} \mathrm{Res}_{\omega}\left(e^{i\omega t}\hat{F}(\omega)\right) = i\sum_{\text{Poles in UHP}} e^{i\omega t}\,\mathrm{Res}_{\omega}\hat{F}(\omega).}$$

B $t<0$. Same approach, but on the contour



$$F(t) = i\sum_{\text{Poles in LHP}} e^{i\omega t}\,\mathrm{Res}_{\omega}\hat{F}(\omega)$$

If this $= 0$ for all $t<0$, there can be <u>no</u> <u>poles</u> <u>in</u> <u>the</u> <u>LHP</u>.

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Interpretation of poles in UHP.

$t \geq 0$: $\quad F(t) = i \sum_{\text{poles}} e^{i\omega t} \, \text{Res}_{\omega} \hat{F}(\omega)$

This must be real. Also, $\hat{F}(-u+iv) = \int e^{-i(-u+iv)t} F(t) dt$

$= \overline{\int e^{iut+vt} F(t) dt}$

$= \overline{\hat{F}(u+iv)}$

So, poles must come in pairs: if $a+ib$ is a pole with residue r_+, $-a+ib$ must also be a pole, with residue, say, r_-.

[No pairing if $a=0$]

$$F(t) = i \sum_{\text{pairs}} \left(e^{i(a_k+ib_k)t} r_{k+} + e^{i(-a_k+ib_k)t} r_{k-} \right)$$

$$+ i \sum_{\text{unpaired}} e^{i(ic_k t)} r_{c_k} \qquad [a, b, c \text{ real}]$$

$\Rightarrow$ r_{c_k} pure imaginary (unpaired), $\overline{r_+} = -r_-$.

$$F(t) = i \sum_{\text{pairs}} e^{i(a_k+ib_k)t} (\alpha_k + i\beta_k) + e^{i(-a_k+ib_k)t} (-\alpha_k + i\beta_k)$$

$$+ i \sum_{\text{unpaired}} e^{i(ic_k t)} (i\gamma_k)$$

[α, β, γ real]

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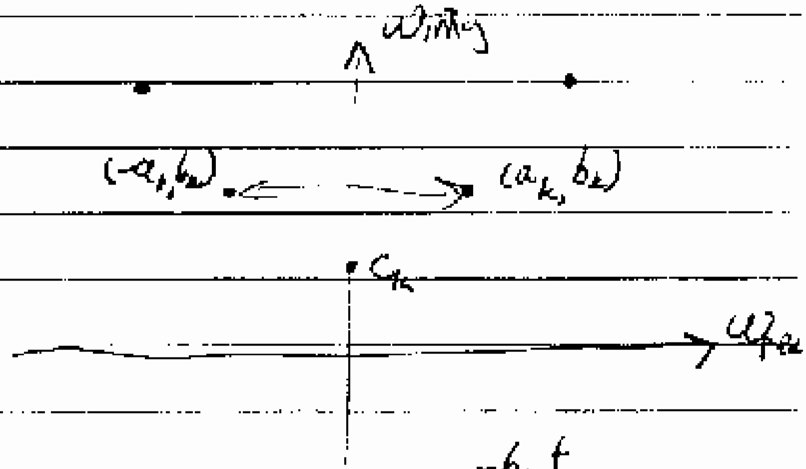
$$F(t) = \sum_{\text{pairs}} e^{-b_k t}\left(\left(-\beta_k + i\alpha_k\right) e^{i a_k t} + c.c.\right)$$

$$+ \sum_{\text{unpaired}} \left(-\gamma_k\right) e^{-c_k t}$$



b_k is dist. of pole from real axis, higher $b_k \rightarrow$ faster decline

Same w/ c_k.

a_k = the frequency under the envelope $e^{-b_k t}$.

α, β, γ show how much these are weighted.

LHP = unstable, near real axis but in UHP → slightly stable

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<u>What happens when systems are combined?</u>

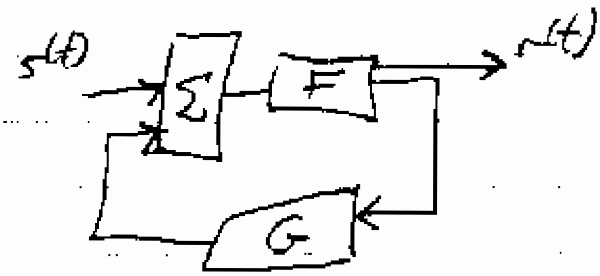
$\hat{F}(\omega) + \hat{G}(\omega)$ [parallel comb]

Poles stay in same place [though may cancel]

$\hat{F}(\omega)\,\hat{G}(\omega)$ [serial comb.]

Poles stay in same place [though may cancel]

$$\frac{\hat{F}(\omega)}{1 - \hat{F}(\omega)\hat{G}(\omega)}$$



Can get new poles if $\hat{F}(\omega)\hat{G}(\omega) = 1$.
This corresponds to $\hat{F}(\omega)\hat{G}(\omega) > 1$ somewhere on the real axis.

Too much gain in the loop!

<u>Not covered here:</u>
- Recovering Re $\hat{F}(\omega)$ from Im $\hat{F}(\omega)$ + vice versa
- " phase $(\hat{F}(\omega))$ from $|\hat{F}(\omega)|$
- Meaning of $\log \hat{F}(\omega)$
- "Minimum phase"

- Systems linear in space + time

- Hilbert transforms
- Instantaneous frequency