1. $$p_j = \sum_{k=0}^{N-1} e^{\frac{2\pi i k j}{N}} \hat{p}_k, \quad q_j = \sum_{m=0}^{N-1} e^{\frac{2\pi i m j}{N}} \hat{q}_m$$

so

$$s_j = \frac{1}{N} \sum_{r=0}^{N-1} p_{j-r}\, q_r$$

$$= \frac{1}{N} \sum_{r,k,m=0}^{N-1} e^{\frac{2\pi i k (j-r)}{N}} e^{\frac{2\pi i m r}{N}} \hat{p}_k \hat{q}_m$$

$$= \frac{1}{N} \sum_{k,m=0}^{N-1} \sum_{r=0}^{N-1} e^{\frac{2\pi i k}{N} j} e^{\frac{2\pi i}{N}(k-m) r} \hat{p}_k \hat{q}_m$$

But $$\sum_{r=0}^{N-1} e^{\frac{2\pi i}{N} a r} = \begin{cases} 0, & a \neq 0 \pmod N \\ N, & \text{otherwise} \end{cases}$$

So RHS $$= \frac{1}{N} \sum_{k,m=0}^{N-1} e^{\frac{2\pi i k}{N} j} \cdot (N\, \delta_{k,m})\, \hat{p}_k \hat{q}_m$$

$$= \frac{1}{N} \cdot N \sum_{k=0}^{N-1} e^{\frac{2\pi i k}{N} j} \hat{p}_k \hat{q}_k = \sum_{k=0}^{N-1} e^{\frac{2\pi i k}{N} j} \hat{p}_k \hat{q}_k$$

Since

$$s_j = \sum_{k=0}^{N-1} e^{\frac{2\pi i k}{N} j} \hat{s}_k, \quad \hat{s}_k = \hat{p}_k \hat{q}_k.$$

2. $$\hat{r}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} r(t)\,dt$$

$$= \int_{-\infty}^{\infty} e^{-i\omega t} s(t+\tau)\,dt$$

$u = t + \tau$

$$= \int_{-\infty}^{\infty} e^{-i\omega(u-\tau)} s(u)\,du$$

$$= e^{i\omega\tau}\int_{-\infty}^{\infty} e^{-i\omega u} s(u)\,du = e^{i\omega\tau}\hat{s}(\omega).$$

3. Method A. $$\hat{q}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}\frac{d}{dt}s(t)\,dt$$

Integrate by parts $$\int_{-\infty}^{\infty} u\,dv = uv\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v\,du, \quad \begin{cases} u = e^{-i\omega t} \\ dv = ds(t) = \frac{ds(t)}{dt}dt \end{cases}$$

$$= e^{-i\omega t}s(t)\Big|_{-\infty}^{\infty} + i\omega\int e^{-i\omega t}s(t)\,dt$$

$$= i\omega\,\hat{s}(\omega).$$

Method B. $$q(t) = \frac{ds(t)}{dt} = \lim_{\tau\to 0}\frac{s(t+\tau)-s(t)}{\tau}$$

So $$\hat{q}(\omega) = \lim_{\tau\to 0}\frac{\hat{r}_{\tau}(\omega)-\hat{s}(\omega)}{\tau}$$ $\hat{r}_{\tau}$ as in prob. 2

$$= \lim_{\tau\to 0}\frac{1}{\tau}\left(e^{i\omega\tau}-1\right)\hat{s}(\omega)$$

$$= i\,\omega\,\hat{s}(\omega).$$

Method C.

$$s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\,\hat{s}(\omega)\,d\omega$$

Differentiate both sides: $$\frac{ds}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} i\omega\, e^{i\omega t}\,\hat{s}(\omega)\,d\omega$$

So

$q(t)$ and $i\omega\hat{s}(\omega)$ are FT pairs

<u>Note</u>

For $q(t) = \frac{d^n}{dt^n}s(t)$, $\hat{q}(\omega) = (i\omega)^n\,\hat{s}(\omega)$

[easiest by Method C]

4. $$\hat{s}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}\,s(t)\,dt$$

$$\frac{d}{d\omega}\hat{s}(\omega) = \int_{-\infty}^{\infty}(-it)\,e^{-i\omega t}\,s(t)\,dt$$

$$\frac{d^n}{d\omega^n}\hat{s}(\omega) = \int_{-\infty}^{\infty}(-it)^n\,e^{-i\omega t}\,s(t)\,dt$$

$$i^n\frac{d^n}{d\omega^n}\hat{s}(\omega)\Big|_{\omega=0} = \int_{-\infty}^{\infty} t^n\,s(t)\,dt$$