FFT-1

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The Fast Fourier Transform Algorithm

(Cooley–Tukey ~ 1960's)

We use f, $\hat{f}$ on integers:

$$\hat{f}(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x)\, e^{-\frac{2\pi i u x}{N}}$$

$$\left(\text{and } f(x) = \sum_{u=0}^{N-1} \hat{f}(u)\, e^{\frac{2\pi i}{N} u x} \right).$$

Seems like N^2 multiply + add op's are needed,
But can we make use of the symmetry of

$$e^{2\pi i \ell / N} \; ?$$

$$\hat{f}\left(u + \tfrac{N}{2}\right) = \frac{1}{N} \sum_{x=0}^{N-1} f(x)\, e^{-\frac{2\pi i u x}{N}}\, e^{-\pi i x}$$

$$(\nabla) \qquad = \frac{1}{N} \left[\sum_{x\,\text{even}} f(x)\, e^{-\frac{2\pi i u x}{N}} - \sum_{x\,\text{odd}} f(x)\, e^{-\frac{2\pi i u x}{N}} \right]$$

Put $f_0(x) = f(2x) \qquad (x = 0, 1, \cdots, N/2 - 1)$

$f_1(x) = f(2x+1) \qquad (x = 0, 1, \cdots, N/2 - 1)$

Then,

$$\hat{f}\left(u + \tfrac{N}{2}\right) = \frac{1}{N} \left(\sum_{x=0}^{\frac{N}{2}-1} f_0(x)\, e^{-\frac{2\pi i u}{N} \cdot 2x} - \sum_{x=0}^{\frac{N}{2}-1} f_1(x)\, e^{-\frac{2\pi i u}{N}(2x+1)} \right)$$

$$= \frac{1}{N} \left(\sum_{x=0}^{\frac{N}{2}-1} f_0(x)\, e^{-\frac{2\pi i u}{N/2} x} - \sum_{x=0}^{\frac{N}{2}-1} f_1(x)\, e^{-\frac{2\pi i u}{N/2} x}\, e^{-\frac{2\pi i u}{N}} \right)$$

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$$\hat{f}\left(u+\frac{N}{2}\right)=\frac{1}{2}\left(\underset{\downarrow\ \frac{N}{2}\text{ points}}{\hat{f}_0(u)}-e^{-\frac{2\pi i}{N}u}\underset{\downarrow\ \frac{N}{2}\text{ points.}}{\hat{f}_1(u)}\right)$$

(1^{st} stage)

Similarly, with a "+" sign in eq. ∇ on p. $\boxed{1}$ ...

$$\hat{f}(u)=\frac{1}{2}\left(\hat{f}_0(u)+e^{-\frac{2\pi i}{N}u}\hat{f}_1(u)\right)$$

This writes the F.T. on N points in terms of the F.T. on N/2 points!

(2^{nd} stage)

$$\hat{f}_0(u)=\frac{1}{2}\left(\hat{f}_{00}(u)+e^{-\frac{2\pi i}{N/2}u}\hat{f}_{01}(u)\right)$$

$$\hat{f}_0\left(u+\frac{N}{4}\right)\ \frac{1}{2}\left(\hat{f}_{00}(u)-e^{-\frac{2\pi i}{N/2}u}\hat{f}_{01}(u)\right)$$

$$\hat{f}_1(u)=\frac{1}{2}\left(\hat{f}_{10}(u)+e^{-\frac{2\pi i}{N/2}u}\hat{f}_{11}(u)\right)$$

$$\hat{f}_1\left(u+\frac{N}{4}\right)=\frac{1}{2}\left(\hat{f}_{10}(u)-e^{-\frac{2\pi i}{N/2}u}\hat{f}_{11}(u)\right)$$

Two parallel FFT's.

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k th stage:

$[\frac{N}{2^{k-1}}$ points$]$ $\qquad$ $[\frac{N}{2^k}$ points$]$

$$\hat{f}_{a_1 a_2 \cdots a_{k-1}}(u) = \frac{1}{2}\left(\hat{f}_{a_1 a_2 \cdots a_{k-1} 0}(u) + e^{-\frac{2\pi i u}{N} 2^{k-1}} \hat{f}_{a_1 \cdots a_{k-1} 1}(u)\right)$$

$$\hat{f}_{a_1 a_2 \cdots a_{k-1}}\left(u + \frac{N}{2^k}\right) = \frac{1}{2}\left(\hat{f}_{a_1 a_2 \cdots a_{k-1} 0}(u) - e^{-\frac{2\pi i u}{N} 2^{k-1}} \hat{f}_{a_1 \cdots a_{k-1} 1}(u)\right)$$

$\hat{f}_{a_1 a_2 \cdots a_{k-1}}$ means a F.T. on $\frac{N}{2^{k-1}}$ points, selected from the original N points by the requirement that the binary expansion of x (the point index) ends with $\cdots a_{k-1} a_{k-2} \cdots a_2 a_1$.

When $k = \log_2 N$, then this a F.T. on 2 points

So the $(\log_2 N)$ th stage is

$$\hat{f}_{a_1 a_2 \cdots a_{k-1}}(u) = \frac{1}{2}\left(f(\underbrace{0 a_{k-1} \cdots a_2 a_1}_{\text{viewed as a binary number}}) \pm f(1 a_{k-1} \cdots a_2 a_1)(-1)^u \right)$$

$u = 0$ or 1

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Summary : $\log_2 N$ stages.

stage r ($r = \log_2 N, \log_2 N - 1, \ldots, 1$):

2^{r-1} separate F.T.'s, each with $\frac{N}{2^{r-1}}$ points.

$\frac{N}{2}$ pairs of calcs - one multiplication, one addition, one subtraction } as complex numbers.

(can neglect the "$\frac{1}{2}$" scale factor at each stage, & apply it once at the end).

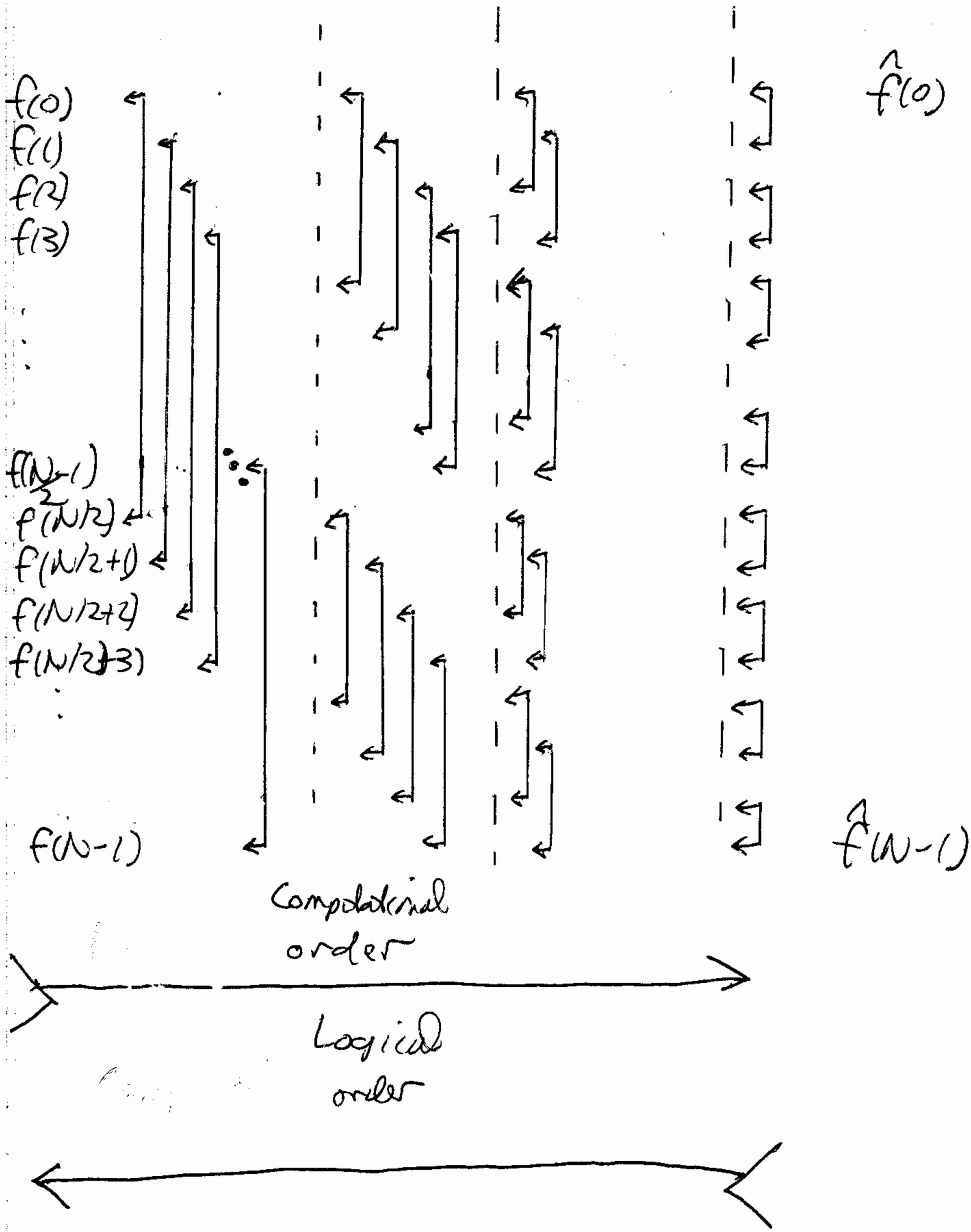
$\frac{3N}{2} \cdot \log_2 N$ complex operations

- however * Fourier components come out in bit-reversed order
  * exponentials need to be calculated (or looked up).

The unshuffling procedure sometimes isn't necessary - but if it is, requires order $N \log_2 N$ steps
Exponentials could be looked up, calculated, or stored

[The direct calculation: $2N$ operations per F.C.
This *may* be more efficient if only a few F.C.'s need to be calculated.]

FFT Overview



Some other considerations:

Say f is real.

$$\text{Then, } \hat{f}(-u) = \hat{f}(N-u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x) e^{-\frac{2\pi i (N-u)}{N}x}$$

$$= \overline{\frac{1}{N}\sum_{x=0}^{N-1} f(x) e^{\frac{2\pi i u}{N}x}}$$

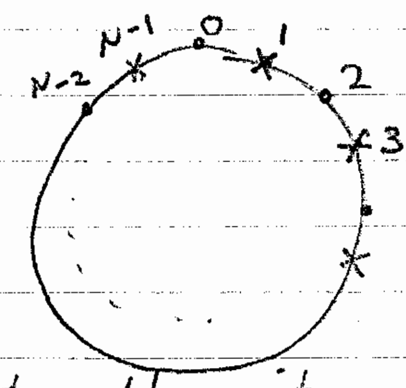
$$= \overline{\hat{f}(u)}$$

(So, at last stage, only have to do $\frac{1}{2}$ as much work)

Similarly, if f is symmetric ($f(x) = f(-x)$), then $\hat{f}(u)$ must be real.

Relation to group representations (sketch)

V = functions on N points.
A function on N points ($0, 1, \cdots, N-1$) is regarded as a pair of functions, one on the <u>even</u> points & one on the <u>odd</u> points.



$V = V_{even} \oplus V_{odd}$. A shift by an <u>even</u> # of points <u>preserves</u> the subspaces & a shift by an <u>odd</u> # of points <u>interchanges</u> them.