① # GROUPS, FIELDS, VECTOR SPACES

General Themes

- avoid the appearance of accidents
- avoid coordinates (or find "natural" coordinates)
- use "good" math. models for data
  - smooth transition betw large + infinite
  - smooth transition betw sampled + continuous

Overview:

- groups - one kind of element, one operation.
- fields - add a second operation
- vector spaces "over" a field
- transformations from one vector space to itself
- groups can be modeled as these transformations
- characterizing these transformations leads to "natural" bases ("coordinates") in the VS
- Fourier theory is a special case of the above

$G = \mathbb{Z}_n$ (under addition) → Discrete Fourier Trans.
$G = \mathbb{R}$ (under addition) → Fourier Trans.
$G =$ rotations of a circle → Fourier Series
. . . .
$G =$ rotation of a sphere → spherical harms.

$\left.\begin{array}{l} G = \text{permutations of } n \text{ items} \\ G = \text{translations of } \mathbb{E}^n \\ G = \text{translations + rotations of } \mathbb{E}^n \end{array}\right\}$ other useful things

Group axioms

G1) Associativity. For all a, b, c $\quad a\circ(b\circ c) = (a\circ b)\circ c$

G2) Identity: There is an element "e" such that, for all a, $\quad a\circ e = a$ and $e\circ a = a$.

G3) Inverses: For all a, there is a corresponding a^{-1} such that

$$a \circ a^{-1} = e$$

$$a^{-1} \circ a = e.$$

Not assumed to be commutative ("Abelian")

May be finite or infinite.

May have other properties (e.g, "nearness" - a topology) (Lie groups)

Examples $\quad \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ under addition

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ with 0 omitted under multiplication

$m \times n$ matrices under addition

$m \times m$ invertible matrices under mult.

group op. is composition $\left\{ \begin{array}{l} \text{rotations of an } n\text{-sphere} \\ \text{rotations of a regular } k\text{-gon} \\ \text{rotations and reflections of a regular } k\text{-gon} \\ \text{permutations of a set of objects} \end{array} \right.$

Which are commutative? Which are finite? Which have a nontrivial topology?

③

Basic properties of identities + inverses

- Only one identity element. If e and f were both identities, then

$$e \circ f = e \qquad \text{since } f \text{ is an identity}$$
$$e \circ f = f \qquad \text{'' } e \text{ '' '' ''}$$

- Inverse is unique. Say $a \circ b = e$. Then

$$\begin{aligned}
G_1 \downarrow \quad a^{-1} \circ (a \circ b) &= a^{-1} \circ e \quad \downarrow G_2 \\
G_3 \downarrow \quad (a^{-1} \circ a) \circ b &= a^{-1} \\
G_2 \downarrow \quad e \circ b &= a^{-1} \\
b &= a^{-1}
\end{aligned}$$

- No element has a "private" identity. For if $a \circ f = a$,

$$\begin{aligned}
a^{-1} \circ (a \circ f) &= a^{-1} \circ a \\
(a^{-1} \circ a) \circ f &= a^{-1} \circ a \\
e \circ f &= e \\
f &= e
\end{aligned}$$

- $(a \circ b)^{-1} = b^{-1} \circ a^{-1}$

$$\begin{aligned}
(a \circ b) \circ (b^{-1} \circ a^{-1}) &= ((a \circ b) \circ b^{-1}) \circ a^{-1} \\
&= (a \circ (b \circ b^{-1})) \circ a^{-1} \\
&= (a \circ e) \circ a^{-1} \\
&= a \circ a^{-1} \\
&= e.
\end{aligned}$$

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Intrinsic properties: the <u>order</u> of an element a is the least (nonzero) integer n for which $\underbrace{a \circ a \cdots \circ a}_{n \text{ times}} = e$. [write $a^n = e$]

a) For finite groups, every element has an order.

Consider a^0 $(=e)$, a^1, a^2, a^3, $\ldots$.
Eventually, there must be a repeat.
If $a^h = a^k$, then, (if $h < k$)
$$(a^h)^{-1} \circ a^h = a^{k-h}$$
$$a^{k-h} = e$$
Order of a must be $\leq k-h$.

b) For finite groups, the order of every element divides the size of the group, "$|G|$".
Let $A = \{a^0, a^1, a^2, \ldots\}$. Size of A is order of a.
A is a <u>subgroup</u> (i.e., a subset of G, but also a group).
Can show $|H|$ divides $|G|$ for any subgroup.

Say $H = \{e, h_1, h_2, \ldots\}$. Write $Hb = \{b, h_1 \circ b, h_2 \circ b, \ldots\}$.
Hb is a "coset" (not nec. a subgroup)
Consider two cosets Hb & Hc. They are either identical or disjoint. If not disjoint, say $h \circ b = h' \circ c$.
Then
$$b = h^{-1} \circ h' \circ c$$
$$h_j \circ b = h_j \circ h^{-1} \circ h' \circ c.$$
So every element in Hb is in Hc. And vice-versa.

⑤

So G is a disjoint union of distinct cosets, all the same size as H.

So $|G|$ is a multiple of $|H|$.

Counting arguments will not work for infinite groups, but averages over the group (sums or integrals) will work.

Relationships among groups [or other structures: fields, VS's]

- Subgroups (defined above)
- Structure-preserving maps

Homomorphism: $\varphi : G \to H$

is a map from group G to group H s.t.

$$\underset{\text{in } G}{\varphi(g_1 \circ g_2)} = \varphi(g_1) \underset{\text{in } H}{\circ} \varphi(g_2)$$

Homomorphism is "onto" H if all members of H are some $\varphi(g)$

↓

Isomorphism - if there is an inverse map $\varphi^{-1} : H \to G$ s.t. $\varphi^{-1}(\varphi(g)) = g$

↓

Automorphism: $\varphi : G \to G$.

Examples

log : a homomorphism from ($\mathbb{R} > 0$, multiplication) to ($\mathbb{R}$, addition)

$n \to 2n$ a homomorphism from ($\mathbb{Z}$, addition) to ($\mathbb{Z}$, addition).

$n \to -n$ "

$z \to e^z$ from ($\mathbb{C}$, addition) to $\mathbb{C} - \{0\}$, multiplication

Parity of a permutation from any permutation group to $\{1, -1\}$ under multip.

Which are onto? Which are isomorphisms? Which are automorphisms?

The kernel of a homomorphism $\varphi: G \to H$ is the set of elements g for which $\varphi(g) = e$ [in H]

The kernel is a subgroup. [Need to show that (a) $\varphi(e) = e$, and (b) $\varphi(g) = e$, then $\varphi(g^{-1}) = e$.]

(a) $\varphi(e) = \varphi(e \circ e) = \varphi(e) \circ \varphi(e)$ So $\varphi(e)$ is the identity for $\varphi(e)$ in H, so $\varphi(e)$ is the id. in H.

(b) $\underbrace{\varphi(g) \circ \varphi(g^{-1})}_{= e \circ \varphi(g^{-1})} = \varphi(g \circ g^{-1}) = \varphi(e) = e$

Objects playing several roles.

The automorphisms form a group, $\mathcal{A}(G)$.
Say φ_1 + φ_2 are both automorphisms.
<u>Define</u>

$$(\varphi_1 \circ \varphi_2)(g) = \varphi_1(\varphi_2(g))$$

[use $\circ$ for the group op in $\mathcal{A}(G)$, write gh within G]

Need to show $\varphi_1 \circ \varphi_2$ is an automorphism.

$$
\begin{aligned}
(\varphi_1 \circ \varphi_2)(gh) &= \varphi_1(\varphi_2(gh)) \\
&= \varphi_1(\varphi_2(g)\varphi_2(h)) && [\varphi_2 \text{ is an auto}] \\
&= \varphi_1(\varphi_2(g))\,\varphi_1(\varphi_2(h)) && [\varphi_1 \text{ is an auto}] \\
&= (\varphi_1 \circ \varphi_2(g))(\varphi_1 \circ \varphi_2(h))
\end{aligned}
$$

Since automorphisms by def. are invertible, there is a φ^{-1}, and, with this definition for $\circ$, $\varphi \circ \varphi^{-1} = e$.

There are special automorphisms - the "inner" automorphisms.
For each group element α in G,
let $\varphi_\alpha(g) = \alpha g \alpha^{-1}$.

Note $\varphi_\alpha(gh) = \alpha g h \alpha^{-1} = \alpha g \alpha^{-1} \alpha h \alpha^{-1} = \varphi_\alpha(g)\varphi_\alpha(h)$
and $\varphi_\alpha^{-1} = \varphi_{\alpha^{-1}}$, so φ_α is invertible.

A model for "change of coordinates".

Now we have a natural map (a homomorphism)

$$\mathrm{Adj}: G \to \mathcal{A}(G) \qquad \text{"adjoint"}$$
$$\mathrm{Adj}(\alpha) = \varphi_\alpha.$$

What is $\underline{\underline{\text{its}}}$ kernel?

$\mathrm{Adj}(\alpha)$ = identity in $\mathcal{A}(G)$ means

$$\varphi_\alpha(g) = g \quad \text{for all } g \in G$$

$$\alpha g \alpha^{-1} = g, \quad \alpha g = g \alpha$$

So, the kernel of the adjoint is the subset of G that commutes with all of G ("the center")

Adj is trivial if G is commutative.
But these groups have nontrivial $\mathcal{A}(G)$.

What is the center of $m \times m$ invertible matrices?
(under mult.)

(9) **Fields**

Add a second operation to the group elements.

First op: "+" (by convention)

Second op: "$\cdot$" (")

Traditionally, field denoted "k"

Field axioms:

+) k: Commutative group under $+$, identity element denoted 0

$\cdot$) $k - \{0\}$: " " " $\cdot$, identity element denoted 1

Relationship of $+$ and $\cdot$:

$$\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$$

$\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$ are fields, $\mathbb{Z}$ is not (why?)

$\mathbb{Z}_n$ ($= \{0, 1, \cdots n-1\}$ interpreted mod n)

may be a field

Need to be sure there are multiplicative inverses.

If $\alpha^{-1} = \beta$, then

$$\alpha\beta \equiv 1 \pmod{n}$$

$$\alpha\beta + hn = 1$$

This is solvable in integers (β, h) if α and n are relatively prime. So, solutions guaranteed if n is prime. Conversely if n is composite, and α shares a divisor > 1 with n, then there's no solution.

This gives us finite fields of size p, any prime.

Can also make finite fields of size p^r ("Galois Fields"), but this is slightly tricky.

$\mathbb{R}$ is "ordered" - a notion of size ($<$)
$\mathbb{C}$ is not ordered.

$\mathbb{C}$ is "algebraically closed": a polynomial $\sum_{k=0}^{r} \alpha_k X^k$

always has a root λ, with $\sum_{k=0}^{r} \alpha_k \lambda^k = 0$

This will help a lot.

$\mathbb{C}$ has an automorphism $\begin{cases} i \to -i, \\ 1 \to 1 \end{cases}$ "complex conjugation",
denoted $\bar{\ }$.

Any $\alpha \in \mathbb{C}$ can be written as

$$\alpha = a + bi \qquad (a, b \in \mathbb{R})$$
$$\bar{\alpha} = \bar{a} + \bar{b}\,\bar{i} \qquad \text{to preserve } +, \cdot$$
$$= a + b\bar{i}$$
$$= a - bi$$

$\pm i$ & 1 are distinguished.

1 is multiplic. identity
$(\pm i)^2$ are only two sol'ns of $x^2 + 1 = 0$.

This idea can be abstracted & generalized!
Say $\sum_{k=0}^{r} \alpha_k X^k$ does not have a root, and does not factor in k. Then formal sums

$\sum_{k=0}^{r-1} \beta_k X^k$ form a field.

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Vector spaces

($k = \mathbb{C}$ unless stated otherwise)

VS axioms: [Scalars $\alpha, \beta, \cdots$ are elements in a field + Vectors, $v, w, \cdots$ which can be added]

- Vectors form a commutative group under addition.
- Scalar multiplication: a rule from $k \times V \to V$ satisfying

$$\alpha(v+w) = \alpha v + \alpha w$$
$$(\alpha+\beta)v = \alpha v + \beta v$$
$$\alpha(\beta v) = (\alpha\beta)v$$

Nothing said about dimension, length, angles, coordinates.

A field can be regarded as a VS over itself
(vector addition = field addition,
scalar mult. = field multiplication.

Ordered n-tuples of field elements form a VS.
Say $v = (v_1, v_2, \cdots, v_n)$
$w = (w_1, w_2, \cdots, w_n)$

Use $v + w = (v_1 + w_1, v_2 + w_2, \cdots, v_n + w_n)$
$\alpha v = (\alpha v_1, \alpha v_2, \cdots, \alpha v_n)$.

Functions on a set ($S = \mathbb{R}$, $\mathbb{C}$, anything) with values in k form a VS

$$(f_1 + f_2)(s) = f_1(s) + f_2(s)$$
$$(\alpha f_1)(s) = \underbrace{\alpha(f_1(s))}_{\text{operations in } k}$$

"Free Vector Space on S"

Linear independence: $v_1, \dots, v_h$ are "linearly independent" if for all nonzero $\alpha_1, \dots, \alpha_h$,
$$\sum \alpha_k v_k \neq 0.$$

Basis. If $v_1, \dots, v_h$ are linearly independent, and

any $v \in V$ can be written as $v = \sum_1^h \alpha_k v_k$, then $\{v_1, \dots, v_h\}$ is a basis for V, & α's are the coordinates of v

Note that if $v = \sum \alpha_k v_k = \sum \beta_k v_k$ then $\alpha_r = \beta_r$

(since $\sum \alpha_k v_k - \sum \beta_k v_k =$

$$\sum (\alpha_k - \beta_k) v_k = 0 \quad)$$

V is then said to be "h-dimensional".

[Alternative bases have the same size]

Def of basis makes sense even if $\{v_1, \dots, v_h, \dots\}$ not finite.

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Unless additional structure is added, no basis set has a privileged role.

Putting together VS's:

V + W VS's, then $V \oplus W$ [direct sum] is a VS with operand $z = (v, w)$ and operations

$$z_1 + z_2 = (v_1 + v_2, w_1 + w_2)$$

V + W VS's: $\mathrm{Hom}(V, W)$, space of all homomorphisms, is also a VS.

To define the VS operations-

say $\varphi_1 : V \to W$ + $\varphi_2 : V \to W$

$$(\varphi_1 + \varphi_2)(v) = \varphi_1(v) + \varphi_2(v)$$
$$(\alpha \varphi_1)(v) = \alpha(\varphi_1(v))$$

If we have a basis $\{v_1, \cdots, v_m\}$ for V and $\{w_1, \cdots, w_n\}$ for W, then

consider

$$\varphi_{ab}\left(\sum_1^m \alpha_k v_k\right) = \alpha_a w_b .$$

This is a homomorphism, and the set $\{\varphi_{ab}\}$ $\begin{array}{l} a=1,\cdots,m \\ b=1,\cdots,n \end{array}$ form a basis for $\mathrm{Hom}(V, W)$.

If we were to write $v = \sum_1^m \alpha_k v_k$ as $\begin{pmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_m \end{pmatrix}$,

$$w = \sum_1^n \beta_k w_k \text{ as } \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$$

then

$$\varphi v = w \text{ : corr to}$$

$$\begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1m} \\ \vdots & & \\ \alpha_{m1} & \cdots & \gamma_{mm} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$$

if $\varphi = \sum_{a=1}^{m} \sum_{b=1}^{n} \gamma_{ba} \varphi_{ab}$.

Special case of Hom (V, W): $W = k$.

Hom (V, k) = "dual" of V (V^*)
= all linear maps from V to k.

Dual of V has same dimension (if dim V finite).
But V + V^* are not the same.
<u>Ex.</u> V = continuous functions on $\mathbb{R}$, integrable
V^* contains, for every g in V, $f \to \int f(x) g(x) dx$.
But V^* also contains $f \to f(x_0)$.
$\delta(x - x_0)$ is not in V.

Even when V is finite, it is a good idea to keep V & V^* separate.

Lights vs mechanisms

OI: difference maps vs discriminants

Another special case: $\mathrm{Hom}(V, V)$

[Our general program is to look for mappings from G to $\mathrm{Hom}(V,V)$ that preserve the group structure].

Recall that "order" is an intrinsic property of a group element $g^{\mathrm{order}(g)} = 1$.

There are intrinsic properties of $\varphi \in \mathrm{Hom}(V,V)$.

determinant

trace

eigenvalues

Need one more way of putting together a VS: "tensor products".

V, W vector spaces, then $V \otimes W$ is a VS composed of formal sums of elements $v \otimes w$, with the rules

$$v \otimes (w_1 + w_2) = (v \otimes w_1) + (v \otimes w_2)$$
$$(v_1 + v_2) \otimes w = (v_1 \otimes w) + (v_2 \otimes w)$$
$$\lambda (v \otimes w) = (\lambda v) \otimes w = v \otimes (\lambda w)$$

If $\{v_1 \cdots v_m\}$ is a basis for V +
$\{w_1, \cdots, w_n\}$ " " " " W then
$\{v_1 \otimes w_1, \cdots, v_m \otimes w_1, \cdots, v_1 \otimes w_n, \cdots, v_m \otimes w_n\}$
is a basis for $V \otimes W$.

But $V \otimes W$ is not the same as $\mathrm{Hom}(V, W)$,
just like V is not the same as V^*.

[but $V \otimes W$ is naturally $\cong$ to $\mathrm{Hom}(V^*, W)$]

Consider $V^{\otimes 2} = V \otimes V$.

For $u = \sum v_k^{(1)} \otimes v_k^{(2)}$, we can write $\sigma(u) = \sum v_k^{(2)} \otimes v_k^{(1)}$.

Also, can define
$$s(u) = \tfrac{1}{2}(u + \sigma(u)) \quad \text{"symmetric"}$$
$$a(u) = \tfrac{1}{2}(u - \sigma(u)) \quad \text{"antisym"}$$

$$\left.\begin{aligned} s(s(u)) &= s(u) \\ s(a(u)) &= a(s(u)) = 0 \\ a(a(u)) &= a(u) \end{aligned}\right\} \text{ all follow from } \sigma(\sigma(u)) = u.$$

e.g.,
$$\begin{aligned} s(a(u)) &= s\left(\tfrac{1}{2}(u - \sigma(u))\right) \\ &= \tfrac{1}{2}\left[s(u) - s(\sigma(u))\right] \\ &= \tfrac{1}{2}\left[\tfrac{1}{2}(u + \sigma(u)) - \tfrac{1}{2}(\sigma(u) + \sigma^2(u))\right] \\ &= 0 \end{aligned}$$

For every pair $v^{(1)}, v^{(2)}$ of vectors in V, we have
vectors $s(v^{(1)} \otimes v^{(2)})$ in $s(V^{\otimes 2})$
and $a(v^{(1)} \otimes v^{(2)})$ in $a(V^{\otimes 2})$.

This generalizes to $V^{\otimes r} = \underbrace{V \otimes \cdots \otimes V}_{r \text{ times}}$

$r=2$:

$$s(u) = \frac{1}{2}\left(v^{(1)} \otimes v^{(2)} + v^{(2)} \otimes v^{(1)}\right)$$
$$a(u) = \frac{1}{2}\left(v^{(1)} \otimes v^{(2)} - v^{(2)} \otimes v^{(1)}\right)$$

General: if $u = v^{(1)} \otimes v^{(2)} \cdots \otimes v^{(r)}$

$$\tau: \begin{array}{l} 1 \to \tau(1) \\ 2 \to \tau(2) \\ \vdots \\ r \to \tau(r) \end{array}$$

$$s(u) = \frac{1}{r!} \sum_{\text{permutations } \tau} \sigma_\tau(u)$$

where $\sigma_\tau(u) = v^{\tau(1)} \otimes \cdots v^{\tau(r)}$

$a(u)$ a bit trickier.

Every permutation has a "sign" = parity of the number of pairwise flips.

For example,

$$\tau: \begin{array}{l} 1 \to 5 \\ 2 \to 3 \\ 3 \to 4 \\ 4 \to 7 \\ 5 \to 1 \\ 6 \to 6 \\ 7 \to 2 \end{array}$$

$= 1 \leftrightarrow 5$ then $2 \leftrightarrow 3$ then $7 \leftrightarrow 2$ then $4 \leftrightarrow 7$

Permutation notation

$(15)(2347) =$

$(15)(23)(72)(47)$

4 swaps, τ is even

$\text{sign}(\tau) = +1$

(173526)

$= (17)(61)(26)(52)(35)$

$$\tau: \begin{array}{l} 1 \to 7 \\ 2 \to 6 \\ 3 \to 5 \\ 4 \to 4 \\ 5 \to 2 \\ 6 \to 1 \\ 7 \to 3 \end{array}$$

$=$ $1 \leftrightarrow 7$ then $6 \leftrightarrow 1$ then $2 \leftrightarrow 6$ then $5 \leftrightarrow 2$ then $3 \leftrightarrow 5$

5 swaps, τ is odd,

$\text{sign}(\tau) = -1$

$$a(u) = \frac{1}{r!} \sum_{\text{permutations } \tau} \text{sign}(\tau) \cdot \sigma_\tau(u).$$

[Need to show that sign (τ) is independent of how τ is decomposed.] Consider

$$P = \prod_{i<j} (\theta_i - \theta_j)$$

Then $\prod_{i<j} (\theta_{\tau(i)} - \theta_{\tau(j)})$ = either P or $-P$.]

Note that a linear transformation L in Hom (V, V) extends naturally to $V^{\otimes r}$, $s(V^{\otimes r})$, $a(V^{\otimes r})$

e.g., $a(L^{\otimes r})(a(v^{(1)} \otimes \cdots \otimes v^{r}) = a(Lv^{(1)} \otimes \cdots \otimes Lv^{(r)})$.

Now let's do a dimension count of $a(V^{\otimes r})$, assuming $\dim V = n$.

$r=1$: trivial. $a(V) = V$.

$r=2$: $\quad a(v_{n_1} \otimes v_{n_2}) = \frac{1}{2}(v_{n_1} \otimes v_{n_2} - v_{n_2} \otimes v_{n_1})$

This is 0 if $n_1 = n_2$. Also,
$a(v_{n_1} \otimes v_{n_2}) = -a(v_{n_2} \otimes v_{n_1})$, so these are not lin. indep.

Conclude $\dim a(V^{\otimes 2}) = \frac{1}{2} n(n-1)$.

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$r=3$: $a(v_{n_1} \otimes v_{n_2} \otimes v_{n_3}) = 0$ if

any of n_1, n_2, n_3 equal. Say $n_a = n_b$.

Then the pair-swap (a,b) inverts $\operatorname{sgn}(\tau)$, but leaves $a(v_1 \otimes \cdots \otimes v_{n_3})$ unchanged.

$\operatorname{Sign}(\tau \cdot (ab)) = \operatorname{sign}(\tau)\operatorname{sign}(ab) = -\operatorname{sign}(\tau)$

Reordering $v_{n_1} \otimes \cdots \otimes v_{n_3}$ ($3!$ orders) yields $\pm$ the same thing.

$\dim a(V^{\otimes 3}) = \frac{1}{3!} n(n-1)(n-2)$.

Similarly $\dim a(V^{\otimes r}) = \frac{1}{r!} n(n-1) \cdots (n-r+1)$.

Let $r=n$. $\dim a(V^{\otimes n}) = 1$.

All elements in $a(V^{\otimes n})$ can be written in the form αz, for some $z \in a(V^{\otimes n})$.

Now given L in $\operatorname{Hom}(V,V)$ $a(L^{\otimes n})(z)$ must be some scalar $\cdot z$. This scalar is the <u>determinant</u> of L.

* We didn't use a basis (just counted it) so $\det(L)$ is basis-independent, i.e. intrinsic.

* If, for any basis $v_1, \cdots, v_n$, $L(v_1), \cdots, L(v_n)$ are <u>linearly dependent</u> then $\det(L) = 0$. And conversely.

det (L) describes how much
$a(v_1 \otimes \cdots \otimes v_n)$ expands, for any $\{v_1, \ldots, v_n\}$.

det (LM): can be thought of as L acting in MV.

det LM is $\dfrac{a(LMv_1 \otimes \cdots \otimes LMv_n)}{a(Mv_1 \otimes \cdots \otimes Mv_n)} \cdot \dfrac{a(Mv_1 \otimes \cdots \otimes Mv_n)}{a(v_1 \otimes \cdots \otimes v_n)}$

so $\det LM = \det L \cdot \det M$.

Characteristic Equation: L in Hom (V, V)

λ in k
I = the identity in Hom (V, V)

$L - \lambda I$ is also in Hom (V, V).

$\det(L - \lambda I)$ = "characteristic equation of L".

This is a polynomial in λ, of degree n.

If $\det(L - \lambda I) = 0$ (i.e., λ a root of the c.e. of L)
then $L - \lambda I$ maps <u>some</u> nonzero vector v to 0
$(L - \lambda I)v = 0 \Rightarrow Lv = \lambda v$
v is an eigenvector, λ is its eigenvalue.

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Plan from here: V: functions of time

Inner product

Self-adjoint operator [applying a filter]

Unitary operator [translation in time]

Unitary representations of a group G: translation in time

$\downarrow$

Decomposes V - a "natural basis"

Coordinates in the natural basis are the

Fourier {coef's, transform}.