# Exam Solutions

**Problem 1**

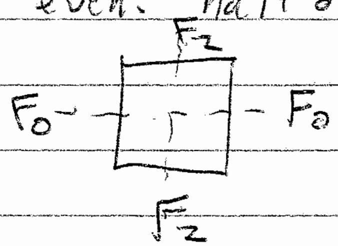
First, set up some notation so that we can write out the elements of $D_n$.

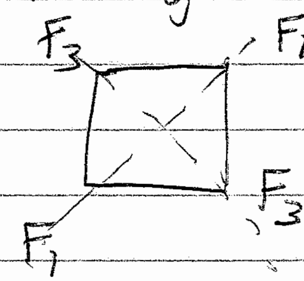
$R_j$: rotation by $\frac{2\pi j}{n}$. As a matrix, $R_j = \begin{pmatrix} \cos\frac{2\pi j}{n} & \sin\frac{2\pi j}{n} \\ -\sin\frac{2\pi j}{n} & \cos\frac{2\pi j}{n} \end{pmatrix}$

Note $R_0 = I$, $R_j R_k = R_k R_j = R_{j+k}$

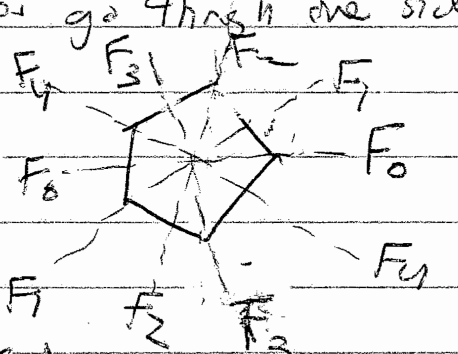
$F_j$: flip on an axis at angle $\frac{\pi j}{n}$.

$n$ even: half of the flips go through 2 sides, half go through 2 corners




$n$ odd: all flips go through one side, one corner




Note $F_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; $F_{2j} = R_j F_0 R_{-j}$; $F_j^2 = I$;

$F_k R_j F_k^{-1} = F_k R_j F_k = R_j^{-1} = R_{n-j}$

$R_0, R_1, \dots, R_{n-1}$ and $F_0, F_1, \dots, F_{n-1}$ are the $2n$ elements

(2)

This suffices to determine the conjugate classes of elements in $D_n$.

Conjugation of a rotation $(R_j)$ with a rotation leads to the same rotation $R_j$

Conjugation of $R_j$ with a flip $(F_k R_j F_k^{-1})$ leads to $R_{-j}$.

Rotations

So the Rotations form conjugate classes as follows:

$$\underbrace{\{R_0\}}_{1\text{ element}} \quad \underbrace{\underbrace{\begin{Bmatrix}R_1\\R_{n-1}\end{Bmatrix}}_{2\text{ elements}} \quad \underbrace{\begin{Bmatrix}R_2\\R_{n-2}\end{Bmatrix}}_{2\text{ elements}} \cdots \underbrace{\begin{Bmatrix}R_{[\frac{n}{2}]-1}\\R_{[\frac{n}{2}]+1}\end{Bmatrix}}_{2\text{ elements}}}_{[\frac{n}{2}]-1 \text{ classes of 2 elements}}$$

and either $R_{\frac{n}{2}}$ ($n$ even) or $\{R_{\frac{n-1}{2}}, R_{\frac{n+1}{2}}\}$ ($n$ odd).

Flips

$F_0$ is conjugate to $F_2, F_4, F_6, \ldots$

$F_1$ is " " $F_3, F_5, F_7, \ldots$

If $n$ is odd, $F_0$ is conjugate to $F_{n+1} = F_1$, and there is one conjugate class $\{F_0, F_1, F_2, \ldots, F_{n-1}\}$ ($n$ elements)

If $n$ is even, the above two series are disjoint, and there are two conjugate classes, each of size $\frac{n}{2}$:

$\{F_0, F_2, \ldots, F_{n-2}\}$

$\{F_1, F_3, \ldots, F_{n-1}\}$

③

$\boxed{1A}$ Now, the group rep's for $D_4$.

Recall:

\# of irred reps = # of conjugate classes

$\sum_i (\text{dim of irred rep})^2$ = size of group

Characters of distinct irred rep's are orthogonal.

See notes below for reasons.

| Conj class: | $I=R_0$ | $\{R_1, R_3\}$ | $R_2$ | $\{F_0, F_2\}$ | $\{F_1, F_3\}$ |
|---|---|---|---|---|---|
| Rep [character table] | Ⓐ | | | | |
| Ⓑ trivial | 1 | 1 | 1 | 1 | 1 |
| Ⓒ parity | 1 | 1 | 1 | -1 | -1 |
| Ⓓ diags | 1 | -1 | 1 | -1 | 1 |
| Ⓔ sides | 1 | -1 | 1 | 1 | -1 |
| Ⓕ matrices | 2 | 0 | -2 | 0 | 0 |

Notes: Ⓐ sum of squares of 5 integers = 8, ⇒ sizes are 1, 1, 1, 1, 2. ⇒ first column

Ⓑ trivial rep maps all elements to unity

Ⓒ parity rep - this is the determinant of the 2×2 matrices, and also the parity of the permutations of the vertices, and also 1 for non-flips, -1 for flips

Ⓓ This is 1 for reps that don't exchange the diagonals, -1 for reps that swap diags

Ⓔ This is Ⓒ⊗Ⓓ. Equivalently, 1 for reps that don't exchange sides, -1 for reps that swap sides

Ⓕ This is the 2×2 matrices on p. 1

(4)

Alternatively, we could have found (B), (C), + (F), and, instead of "guessing" (B) + (C), deduced it from (F)⊗(F).

(F)⊗(F) - has characters 4, 0, 4, 0, 0

so (F)⊗(F) contains 4 irred rep's ($\frac{1}{8}(4^2+4^2)=4$).

This forces the values in rows (C) + (D).

$\boxed{1B}$ Now, $D_5$. Let $\alpha = e^{\frac{2\pi i}{5}}$

| Rep | $I=R_0$ | $\{R_1, R_4\}$ | $\{R_2, R_3\}$ | $\{F_0, F_1, F_2, F_3, F_4\}$ |
|---|---|---|---|---|
| (A) | | | | |
| (B) Trivial | 1 | 1 | 1 | 1 |
| (C) Parity | 1 | 1 | 1 | -1 |
| (D) matrices | 2 | $\alpha+\alpha^4$ | $\alpha^2+\alpha^3$ | 0 |
| (G) | 2 | $\alpha^2+\alpha^3$ | $\alpha+\alpha^4$ | 0 |

(A)': Sum of squares of 4 integers $=10 \Rightarrow 1^2+1^2+2^2+2^2$

(B) as before.

(C)' as before but all permutations are even.
So this is just the det. of the $2\times 2$ matrices.

(D)' The $2\times2$ matrices

(G): look at (D)'⊗(D)', characters are (Z) $[4, (\alpha+\alpha^4)^2, (\alpha^2+\alpha^3)^2, 0]$

Noting that $\alpha^5=1$ this is (Z) $[4, 2+\alpha^2+\alpha^3, 2+\alpha+\alpha^4, 0]$

(Z)·Trivial $=\frac{1}{10}(4+2(2+\alpha^2+\alpha^3)+2(2+\alpha+\alpha^4))=\underline{1}$

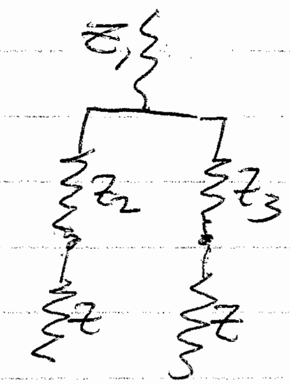
(Z)·Parity also $=\underline{1}$. So (Z) contains trivial (B) + parity (C).

(G) = (Z) − (B) − (C) $= [2 \quad \alpha^2+\alpha^3 \quad \alpha+\alpha^4 \quad 0]$, irred. since $|G|=\underline{1}$.

Problem 2

I had intended this to be interpreted as the $Z_i$'s being impedances. Impedances in series combine by addition; Impedances in parallel combine as $\dfrac{1}{\frac{1}{Z_A}+\frac{1}{Z_B}}$. [Consequence of $V = IZ$]

The network can be written $Z =$




So $$Z = Z_1 + \cfrac{1}{\cfrac{1}{Z+Z_2} + \cfrac{1}{Z+Z_3}}$$

$$Z = Z_1 + \frac{(Z+Z_2)(Z+Z_3)}{2Z+Z_2+Z_3}$$

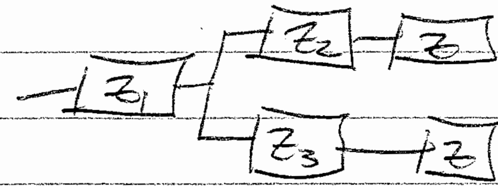
$$Z^2 - 2ZZ_1 - Z_1Z_2 - Z_1Z_3 - Z_2Z_3 = 0$$

$$Z = Z_1 \pm \sqrt{Z_1^2 + Z_1(Z_1+Z_2+Z_3)}$$

only (+) root is meaningful

$$Z = Z_1 + \sqrt{2Z_1^2 + Z_1(Z_2+Z_3)}$$

If you interpret the $Z$'s as systems:

$Z =$  , at least formally,

$$\tilde{Z} = \tilde{Z}_1 \left( \tilde{Z} \tilde{Z}_2 + \tilde{Z} \tilde{Z}_3 \right)$$

solves only for $\tilde{Z} = 0$.

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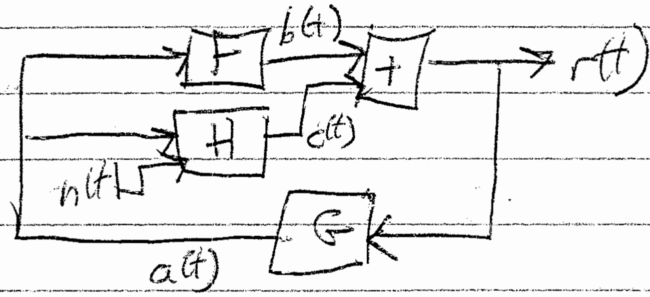
Problem 3




$a(t) =$ output of $G$ ; $\tilde{a}(\omega) = \hat{G}(\omega)\tilde{r}(\omega)$

$b(t) =$ output of $F$ ; $\tilde{b}(\omega) = \hat{F}(\omega)\tilde{a}(\omega) = \hat{F}(\omega)\hat{G}(\omega)\tilde{r}(\omega)$

$c(t) =$ output of $H$ ; $\tilde{c}(\omega) = \hat{H}(\omega)(n(t) + a(t))$

$\hat{H}(\omega)\tilde{n}(\omega) + \hat{G}(\omega)\hat{H}(\omega)\tilde{r}(\omega)$

$r(t) = b(t) + c(t) \Rightarrow \tilde{r}(\omega) = \hat{F}(\omega)\hat{G}(\omega)\tilde{r}(\omega) + \hat{H}(\omega)\hat{G}(\omega)\tilde{r}(\omega) + \hat{H}(\omega)\tilde{n}(\omega)$

$$\tilde{r}(\omega) = \frac{\hat{H}(\omega)}{1 - \hat{G}(\omega)(\hat{H}(\omega) + \hat{F}(\omega))}\tilde{n}(\omega)$$

$$\text{Power spec of } r(t) = |\tilde{r}(\omega)|^2 = \left|\frac{\hat{H}(\omega)}{1 - \hat{G}(\omega)(\hat{F}(\omega) + \hat{H}(\omega))}\right|^2 \hat{N}(\omega).$$

$(\hat{N}(\omega) = |\tilde{n}(\omega)|^2)$

⑧

**Problem 4** A — Constraints are on $x_1, x_2, x_3, x_1^2, x_2^2, x_3^2,$
$x_1x_2, x_2x_3$.

So M.E. dist is of form

$$K \exp\{a_1x_1 + a_2x_2 + a_3x_3 + b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + b_{12}x_1x_2 + b_{23}x_2x_3\}$$

$a_1 = a_2 = a_3 = 0$ by symmetry.
We showed that, for

$$\rho(\vec{x}) = \frac{(\det M)^{1/2}}{\sqrt{2\pi}^{d}} e^{-x^T M x/2},$$

the covariance matrix is $M^{-1}$.

So, we need to find an $M$ of the form $\begin{pmatrix} \circ & \circ & 0 \\ \circ & \circ & \circ \\ 0 & \circ & \circ \end{pmatrix}$

[$\circ$ represents something nonzero] or zero
for which
$$M^{-1} = \begin{pmatrix} 1 & \alpha & \gamma \\ \alpha & 1 & \beta \\ \gamma & \beta & 1 \end{pmatrix}.$$

$$M = \begin{pmatrix} 1 & \alpha & \gamma \\ \alpha & 1 & \beta \\ \gamma & \beta & 1 \end{pmatrix}^{-1}. \quad M_{13} = \frac{\begin{vmatrix} \alpha & 1 \\ \gamma & \beta \end{vmatrix}}{\det M} \quad \text{[by minors]}$$

so $M_{13} = 0$ only if $\begin{vmatrix} \alpha & 1 \\ \gamma & \beta \end{vmatrix} = 0$, i.e., $\gamma = \alpha\beta$.

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Part B.

Here, $M^{-1} = \begin{pmatrix} 1 & 0 & h \\ 0 & 1 & 0 \\ h & 0 & 1 \end{pmatrix}$. $\det M^{-1} = 1 - h^2$

So, $M = (M^{-1})^{-1} = \dfrac{1}{\det M^{-1}} \begin{pmatrix} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ h & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ h & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & h \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & h \\ h & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ h & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & h \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & h \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \end{pmatrix}$

$$M = \frac{1}{1-h^2} \begin{pmatrix} 1 & 0 & -h \\ 0 & 1-h^2 & 0 \\ -h & 0 & 1 \end{pmatrix} \qquad \det M = \frac{1}{1-h^2}$$

So $p(\vec{x}) = \dfrac{1}{(\sqrt{2\pi})^3} \dfrac{1}{\sqrt{1-h^2}} e^{-\frac{1}{2(1-h^2)}\left(x_1^2 + (1-h^2) x_2^2 + x_3^2 + (-2h) x_1 x_3\right)}$

This works for $h > 0$ or $h < 0$; $\langle x_1 x_3 \rangle = h$.

(10)

**Problem 5** A. (k=2)

| Inputs \ Outputs | b=1 | b=2 |
|---|---|---|
| a=1 | $r_{11}$ | $r_{12}$ |
| a=2 | $r_{21}$ | $r_{22}$ |

Input signals occur with probability $\frac{1}{k} \Rightarrow r_{11}+r_{12}=\frac{1}{2}$, $r_{21}+r_{22}=\frac{1}{2}$

Output signals occur with prob $\frac{1}{k} \Rightarrow r_{11}+r_{21}=\frac{1}{2}$, $r_{12}+r_{22}=\frac{1}{2}$

Let $r_{11}=x$. Table is $\begin{array}{c|c} x & \frac{1}{2}-x \\ \hline \frac{1}{2}-x & x \end{array}$, to satisfy above 4 eq'ns.

Ideal Observer performance is:

If $b=1$ is observed (prob. $q_1=r_{11}+r_{21}$), then IO will

(1) guess $a=1$ if $r_{11}>r_{21}$, [occurs $r_{11}$ of the time]

(2) guess $a=2$ if $r_{21}>r_{11}$ [" $r_{21}$ of the time]

If $b=2$ is observed (prob. $q_2=r_{12}+r_{22}$), then IO will

(3) guess $a=1$ if $r_{12}>r_{22}$ [occurs $r_{12}$ of the time]

(4) guess $a=2$ if $r_{22}>r_{12}$ [occurs $r_{22}$ of the time]

Say $x>\frac{1}{4}$. Frac correct $= 2x$ [$x$ from (1) or (3)]

Say $x<\frac{1}{4}$. Frac correct is $2(\frac{1}{2}-x)$ [$\frac{1}{2}-x$ from (2) or (4)]

Frac correct $= \max(2x, 1-2x) = f$.

So, $x=\frac{f}{2}$ or $(1-f)/2$. Table is

| | b=1 | b=2 |
|---|---|---|
| a=1 | $\frac{f}{2}$ | $\frac{1-f}{2}$ |
| a=2 | $\frac{1-f}{2}$ | $\frac{f}{2}$ |

OR

| | b=1 | b=2 |
|---|---|---|
| a=1 | $\frac{1-f}{2}$ | $\frac{f}{2}$ |
| a=2 | $\frac{f}{2}$ | $\frac{1-f}{2}$ |

5B Table will look like

| | $b=1$ | $b=2$ | $b=3$ |
|---|---|---|---|
| $a=1$ | $r_{11}$ | $\cdot$ | $\cdot$ |
| $a=2$ | $\cdot$ | $r_{22}$ | $\cdot$ |
| $a=3$ | $\cdot$ | $\cdot$ | $r_{33}$ |

with (A) $r_{11}, r_{22}, r_{33}$ largest in their respective columns,

(B) row sums $= \frac{1}{3}$.

(C) col sums $= \frac{1}{3}$.

and $f = r_{11} + r_{22} + r_{33}$. [Other arrangements also possible, via permuting columns]

Minimum info: off-diagonal terms equal. (Otherwise, could "mix" with another table, & reduce info.)

| | $b=1$ | $b=2$ | $b=3$ |
|---|---|---|---|
| $a=1$ | $\frac{f}{3}$ | $\frac{1-f}{6}$ | $\frac{1-f}{6}$ |
| $a=2$ | $\frac{1-f}{6}$ | $\frac{f}{3}$ | $\frac{1-f}{6}$ |
| $a=3$ | $\frac{1-f}{6}$ | $\frac{1-f}{6}$ | $\frac{f}{3}$ |

Observer is correct $f = 3\left(\frac{f}{3}\right)$ of the time. When wrong, observer has a 50:50 chance of being right on 2nd guess.

Max info: off-diag terms are maximally unequal.

| | $b=1$ | $b=2$ | $b=3$ |
|---|---|---|---|
| $a=1$ | $\frac{f}{3}$ | $\frac{1-f}{3}$ | $0$ |
| $a=2$ | $0$ | $\frac{f}{3}$ | $\frac{1-f}{3}$ |
| $a=3$ | $\frac{1-f}{3}$ | $0$ | $\frac{f}{3}$ |

Here, the observer will be correct $f = 3\left(\frac{f}{3}\right)$ of the time - but, when wrong, will always get it right on the second guess.