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# Power Spectra - ctd.

Previous HW $\Rightarrow$ P.S. of a Poisson process with renewal density $\lambda e^{-\lambda t}$ is $P_X(\omega) = \lambda$.

What about a non-Poisson renewal process?

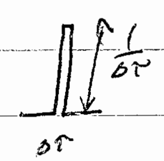
Think of $x(t) = \sum \delta(t - \tau_i)$, $\tau_i$ governed by renewal density $p(x)$.

$$P_X(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau} C_X(\tau)\,d\tau \text{ , where}$$

$$C_X(\tau) = \langle (x(t) - x_0)(x(t+\tau) - x_0) \rangle_t ,$$

$$x_0 = \text{mean of } x(t) = \frac{1}{i\tilde{p}'(0)} = \lambda$$

To show that the P.S. makes sense for a point process, consider replacing the $\delta$-functions by




Will take $\Delta\tau \to 0$, and keep $\int_0^{\Delta\tau} p(t)\,dt$ small.

Calculate $C_X(\tau)$ in 3 regimes: $|\tau| < \Delta\tau$ (1), $\tau > \Delta\tau$ (2), $\tau < -\Delta\tau$ (3)

Also, rewrite $C_X(\tau)$ as

$$\langle x(t)\,x(t+\tau) \rangle_t - x_0^2 .$$

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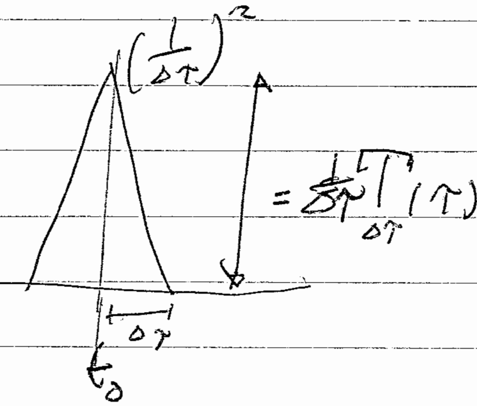
① For $|\tau| < \Delta T$, $\langle x(t)\, x(t+\tau)\rangle \neq 0$ only if $x(t) \neq 0$.

$\int_0^{\Delta T} \rho(t)\,dt$ small means that we don't have to consider two separate events

Given an event at time $t_0$,
$\langle x(t)x(t+\tau)\rangle$ looks like




This is also a peak of area $\frac{1}{\Delta T}$

$$\left[\left(\tfrac{1}{2}\right)(2\Delta T)\left(\frac{1}{(\Delta T)^2}\right) = \frac{1}{\Delta T}\right]$$

So ① contributes $\frac{1}{\Delta T}\delta(\tau) \cdot$ [probability that $x(t) \neq 0$]
$= \lambda\,\delta(\tau)$.

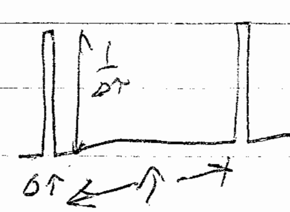
[[The $x_0^2$-term is smaller (by $\Delta T$)]]

A contribution of $\lambda\,\delta(\tau)$ to $C_x(\omega)$ is a contribution of

$\lambda$ to $P_x(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau}\left[\lambda\,\delta(\tau) + ② + ②'\right] d\tau$.

② $\tau > \Delta T$.
Given an event at time zero, the contribution to
$\langle x(t)x(t+\tau)\rangle$ from a next spike at time $\tau$ is

$$\frac{1}{\Delta T}\rho(\tau) * \Pi_{\Delta T}(\tau)$$



Similarly, contribution from $n^{th}$ spike at time $\tau$ is

$$\frac{1}{\Delta\tau} \underbrace{p(\tau) * \cdots * p(\tau)}_{n \text{ times}} * \prod_{\Delta\tau}(\tau)$$

So, contrib to $P_x(\omega) = \hat{C}_x(\omega)$ is given by

$$\lambda \left[ \tilde{\prod_{\Delta\tau}}(\omega) \left( \hat{p}(\omega) + \hat{p}(\omega)^2 + \hat{p}(\omega)^3 + \cdots \right) \right]$$

$\lambda\Delta\tau$ = prob of an event includes $t=0$, at resolution $\Delta\tau$.

As $\Delta\tau \to 0$, $\hat{\prod_{\Delta\tau}}(\omega) \to 1$ ($\prod_{\Delta\tau} \to$ a delta-function]

So ② $\to$ a contribution of $\dfrac{\lambda \hat{p}(\omega)}{1-\hat{p}(\omega)}$,

except possibly at $\omega=0$ - since we left off the DC - term on p. 15

②': $\tau < \Delta\tau$. $\frac{1}{\Delta\tau} p(-\tau) * \cdots p(-\Delta\tau) \prod_{\Delta\tau}(\tau)$ is

the contribution of the $n^{th}$ spike at time $\tau$, given a spike at time 0.

So ②' contributes $\dfrac{\lambda \hat{p}(\omega)}{1-\hat{p}(\omega)}$.

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So, except at $\omega = 0$,

$$P_X(\omega) = \overset{(1)}{\lambda} + \overset{(2)}{\frac{\lambda \hat{p}(\omega)}{1-\hat{p}(\omega)}} + \overset{(2')}{\frac{\lambda \overline{\hat{p}(\omega)}}{1-\overline{\hat{p}(\omega)}}}$$

$$= \frac{\lambda}{|1-\hat{p}(\omega)|^2}\Big[1 - \hat{p}(\omega) - \overline{\hat{p}(\omega)} + |p(\omega)|^2$$
$$+ \hat{p}(\omega) - |p(\omega)|^2$$
$$+ \overline{\hat{p}(\omega)} - |p(\omega)|^2\Big]$$

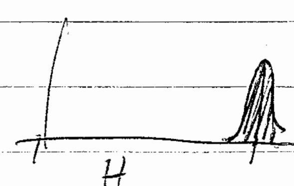
$$= \lambda \frac{1 - |\hat{p}(\omega)|^2}{|1-\hat{p}(\omega)|^2}$$

Calc is actually OK at $\omega = 0$ since $c_X(\tau)$ is well-behaved as $\tau \to \infty$.

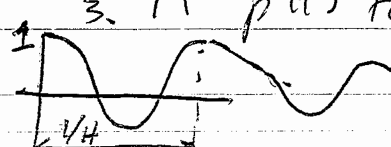
Comments: 1. As $\omega \to \infty$, $\hat{p}(\omega) \to 0$, so $P_X(\omega) \to \lambda$.

2. If $p(t) = \lambda e^{-\lambda t}$, $\hat{p}(\omega) = \frac{1}{1+i\omega/\lambda}$, $P_X(\omega) = \lambda$.

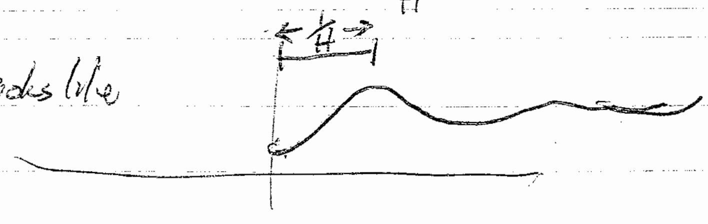
3. If $p(t)$ looks like




$\hat{p}(\omega)$:




then $P_X(\omega)$ looks like



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Near $\omega = 0$, recall

$$\hat{\rho}(\omega) = 1 - ia\omega - \frac{\omega^2 b}{2} \cdots \quad \text{where}$$

$$a = i\tilde{\rho}'(0) = \langle t \rangle = \frac{1}{\lambda}$$
$$b = -\tilde{\rho}''(0) = \langle t^2 \rangle$$

$$1 - |\hat{\rho}(\omega)|^2 = \left(1 - ia\omega - \frac{b\omega^2}{2} \cdots\right)\left(1 + ia\omega - \frac{b\omega^2}{2} \cdots\right)$$

$$= 1 + (a^2 - b)\omega^2 + O(\omega^3)$$

$$|1 - \hat{\rho}(\omega)|^2 = a^2\omega^2 + O(\omega^3)$$

$$\lim_{\omega \to 0} P_x(\omega) = \lim_{\omega \to 0} \lambda \frac{(1 - |\hat{\rho}(\omega)|^2)}{|1 - \hat{\rho}(\omega)|^2} = \lambda \frac{b - a^2}{a^2}$$

$$= \lambda^3 \langle (t - \langle t \rangle)^2 \rangle$$

## Indirect strategies for calculating a power spectrum

$$s(t) \longrightarrow \boxed{\;L\;} \longrightarrow r(t)$$

How about fitting a parametric model for L?

"Moving Average": $\quad r(t) \to \cdots y_1, y_2, \cdots y_n, \cdots$

$\qquad\qquad\qquad\qquad s(t) \to \cdots x_1, x_2, \cdots x_n, \cdots$

Moving Average model of order Q:

$$y_n = \sum_{k=0}^{Q} b_k x_{n-k}, \quad X\text{'s indep. drawn from unit Gaussian}$$

Let $Y_M(\omega) = \dfrac{1}{\sqrt{M}} \displaystyle\sum_{-M/2}^{M/2} y_m e^{-i\omega m}$ [sampling at unit times]

$$P_y(\omega) = \lim_{M\to\infty} |Y_M(\omega)|^2$$

$$Y_M(\omega) = \frac{1}{\sqrt{M}} \sum_{m=-\frac{M}{2}}^{M/2} y_m e^{-i\omega m} = \frac{1}{\sqrt{M}} \sum_{m} \sum_{k=0}^{Q} b_k e^{-i\omega k} x_{m-k} e^{-i\omega(m-k)}$$

For large M (neglecting end-effects), $B(\omega) = \displaystyle\sum_{k=0}^{Q} b_k e^{-i\omega k}$

$$Y_M(\omega) = B(\omega) X_M(\omega) \quad \left[ X_M(\omega) = \frac{1}{\sqrt{M}} \sum_{m=-\frac{M}{2}}^{\frac{M}{2}} x_m e^{-i\omega m} \right]$$

$$\langle X_M(\omega)\overline{X_M(\omega)}\rangle = \frac{1}{M}\sum_{m,n}\langle x_m x_n e^{-i\omega m} e^{i\omega n}\rangle$$

Assuming $x_m$'s indep drawn from a unit Gaussian,

$$\langle x_m x_n\rangle = \begin{cases} 1, & m=n \\ 0, & \text{otherwise}\end{cases}$$

[we could have taken x's drawn from a non-Gaussian dist, provided $\langle x_m x_n\rangle = \delta_{mn}$ ]

So $$P_y(\omega) = \lim_{M\to\infty}\langle |B(\omega)X_M(\omega)|^2\rangle$$
$$= |B(\omega)|^2 = \left|\sum b_k (e^{-i\omega})^k\right|^2.$$

How to find the b's?

$$\langle y_n y_{n+r}\rangle = \left\langle\left(\sum_{k=0}^{Q} b_k x_{n-k}\right)\left(\sum_{l=0}^{Q} b_l x_{n+r-l}\right)\right\rangle$$

Since x's are iid, the only cross terms that contribute are those with $n-k = n+r-l$,
or $k = l-r$

E.g., $Q=2$: $\langle y_n^2\rangle = b_0^2 + b_1^2 + b_2^2$
$\langle y_n y_{n+1}\rangle = b_0 b_1 + b_1 b_2$
$\langle y_n y_{n+2}\rangle = b_0 b_2$

This is nothing more than estimating the cross-covariance; we haven't gained anything.

Coef's of $(e^{i\omega r} + e^{-i\omega r})$ in $|B(\omega)|^2$ will be these quantities.

Problem: for lags $> Q$, we require $\langle y_n y_{n+r} \rangle = 0$, counter intuitive.

Another view of the problem: (related to max entropy)
We have written (say)

$$y_Q = b_Q x_0 + b_{Q-1} x_1 + \cdots + b_0 x_Q$$
$$y_{Q+1} = \qquad b_Q x_1 + b_{Q-1} x_2 + \cdots + b_0 x_{Q+1}$$

etc., i.e., $\vec{y} = L\vec{x}$ ; $L$ looks like $\begin{bmatrix} \times\times\times & & & 0 \\ & \times\times\times & & \\ & & \times\times\times & \\ 0 & & & \times\times\times \end{bmatrix}$

so

$$p(\vec{y})\,d\vec{y} = \frac{1}{\sqrt{2\pi}^{\,Q}}\, e^{-|\vec{x}|^2/2}\, d\vec{x}$$

$$= \frac{1}{\sqrt{2\pi}^{\,Q}}\, e^{-|L^{-1}\vec{y}|^2/2}\, \frac{d\vec{y}}{|\det L|}$$

$L^{-1}$ has many nonzero elements beyond this band of width $(Q+1)$
[$L^{-1}$ typically has no zeros]
so the constructed signal $y$ from the MA model
is NOT max ent given the constraints $\langle y_n y_{n+r} \rangle$.

To fix this:

Autoregressive (AR) model

$$y_n = x_n + \sum_{k=1}^{S} a_k y_{n-k}, \quad x\text{'s indep + Gaussian}$$

Again with $Y_M(\omega) = \frac{1}{\sqrt{M}} \sum_{-M/2}^{M/2} y_m e^{-i\omega m}$

and $A(\omega) = \sum_{k=1}^{S} a_k e^{i\omega k}$

$$Y_M(\omega) = X_M(\omega) + A(\omega) Y_M(\omega)$$

$$P_y(\omega) = \lim_{M\to\infty} |Y_M(\omega)|^2 = \frac{|X_M(\omega)|^2}{|1 - A(\omega)|^2}$$

Since we now have $x_n = y_n - \sum_{k=1}^{S} a_n y_{n-k}$,

we have

$$\vec{x} = G\vec{y}, \text{ and}$$

$$p(\vec{y})\,d\vec{y} = \frac{1}{(\sqrt{2\pi})^S} e^{-|G\vec{y}|^2/2} |\det G|\, d\vec{y}$$

$GG^T$ has 0's where we didn't have constraints on $\langle y_n y_{n+k} \rangle$.
So this is maxent.

But how to calculate the a's?

Let $R = \frac{1}{N}\sum_{n=1}^{N}\left(y_n - \sum_{k=1}^{S} a_k y_{n-k}\right)^2, = \frac{1}{N}\sum x_n^2$

minimize $R$. This minimizes the unmodeled noise $x$, but does not ensure that they are i.i.d. [Can increase $S$ to see if there is a "significant" improvement of fit]

$R$ is a quadratic f'n of the a's, so the minimization will be done by a set of linear equations, namely

$$\frac{\partial R}{\partial a_m} = 0. \quad \text{"Yule - Walker" equations}$$

$m^{th}$ equation:

$$a_1 \langle y_{n-1} y_{n-m} \rangle + \cdots + a_S \langle y_{n-S} y_{n-m} \rangle = \langle y_n y_{n-m} \rangle$$

Need a policy for how to treat ends. But ignoring ends,

$$\sum_{k=1}^{S} a_m \langle y_{n-k} y_{n-m} \rangle = \langle y_n y_{n-m} \rangle$$

depends only on $k-m$.

Note $R = \langle x_n^2 \rangle$, so

$$P_y(\omega) = \frac{R}{|1 - A(\omega)|^2}$$

* Can also have mixed MA–AR models

$$y_n = \sum_{k=0}^{Q} b_k x_{n-k} + \sum_{k=1}^{S} a_k y_{n-k}$$

$$P_y(\omega) = \frac{|B(\omega)|^2}{|1 - A(\omega)|^2}.$$

* These procedures generalize to multichannel scenarios, e.g.,

$$y_{n,c} = x_{n,c} + \sum_{k=1}^{S} \sum_{c'} a_{k,c,c'} \, y_{n-k,c'}$$

"MLAR" (+ ML-MA, etc.)

* stability: when is

$$y_n = x_n + \sum_{k=1}^{S} a_k y_{n-k}$$ stable?

View as

$$y_n - \sum_{k=1}^{S} a_k y_{n-k} = x_n$$, a linear difference equation.

Seek solutions $y_n = z^n$ to the homogeneous equation

$$y_n - \sum_{k=1}^{S} a_k y_{n-k} = 0.$$

We anticipate $S$ linearly independent solutions, for $z = z_1, \dots, z_S$.

| | $y_n$ | | | | | |
|---|---|---|---|---|---|---|
| time: | $n = -S+1 \ \dots$ | $-1$ | $0$ | $1$ | $2$ | $\dots$ |
| $h^{th}$ solution: | $z_h^{-S+1}$ | $z_h^{-1}$ | $1$ | $z_h$ | $z_h^2$ | $\dots$ |

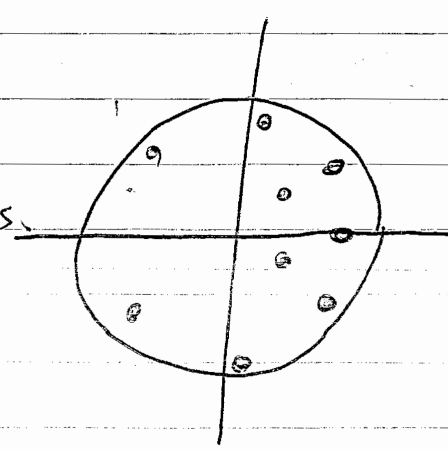
An $S \times S$ matrix. If full rank, then some combination of rows $= \boxed{0 \ 0 \ \dots \ 0 \ 1}$, so we've made a solution of the inhomogeneous eq. starting with a unit pulse at time 0

Solutions to $y_n - \sum_{k=1}^{S} a_k y_{n-k} = 0$

are roots of $z^n - \sum_{k=1}^{S} a_k z^{n-S} = 0.$

If all $|z| < 1$, then stable.
If any $|z| > 1$, then unstable.

Must occur in complex conj pairs.



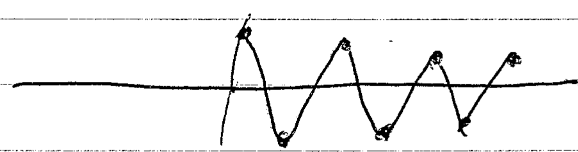
Can rewrite

$$P_y(\omega) = \frac{R}{|1 - A(\omega)|^2} = \frac{R}{\prod_{h=1}^{5} \left| \frac{e^{j\omega}}{z_h} - 1 \right|^2}.$$

The nearness of $z_h$ to the rim indicate the height of the spectral peak

The frequency is given by the phase of $z_h$.
(recall the sampling rate = frequency 1)

So, if there are roots close to $z = -1$, sampling is inadequate
(This is the Nyquist limit, & corresponds to
a component in the AR model looking like



in response to $x_0 = 1$ )

MAIN DRAWBACK : spectral shape must be "similar" to model $\frac{R}{|1 - A(\omega)|^2}$ ; line widths forced by model.

MAIN ADVANTAGE: If spectral shape is OK, then estimates of center frequency not limited by a "frequency bin"