①

Power Spectra 1-14 Ans.

1. $P_x(\omega) = \lim_{T\to\infty} \frac{1}{T} \langle |\int_0^T x(t) e^{-i\omega t} dt|^2 \rangle$

Consider $\int_0^T x(t)\, e^{-i\omega t} dt = \sum_k z_k$, where

$$z_k = \int_{k\Delta T}^{(k+1)\Delta T} x(t)\, e^{-i\omega t} dt.$$

Each z_k is independent so $\langle (\sum_{k=0}^{\frac{T}{\Delta T}-1} z_k)(\sum_{k=0}^{\frac{T}{\Delta T}-1} \bar{z}_k) \rangle$

$$= \frac{T}{\Delta T} \langle |z_1|^2 \rangle$$

As $\Delta T \to 0$: $\langle |z_1|^2 \rangle =$ with probability $\lambda \Delta T$

$= 0$ " " $1 - \lambda \Delta T$

(actually $(\lambda T)^2$)

$$P_x(\omega) = \lim_{T\to\infty} \frac{1}{T} \cdot \frac{T}{\Delta T} \cdot \left(\lambda \Delta T + (1 - \lambda \Delta T)(\lambda T)^2 \right\rangle$$

$= \lambda.$

Bispectrum: same calc. $\left(\int_0^{\Delta T} x(t) e^{-i\omega_1 t} dt\right)\left(\int_0^{\Delta T} x(t) e^{-i\omega_2 t} dt\right) \cdot$

$$\left(\int_0^{\Delta T} x(t)\, e^{+i(\omega_1+\omega_2) t} dt \right)$$

$=$, with prob. $\lambda \Delta T$.

$= (-\lambda \Delta T)^3$ with prob. $1 - \lambda \Delta T$.

$B_x(\omega_1, \omega_2) = \lambda.$

②

Power Spectra, 1-14 Ans.

2. From ①:

$$P_y(\omega) = |\hat{G}(\omega)|^2 \hat{P}_X(\omega)$$

so $P_y(\omega) = \lambda |\hat{G}(\omega)|^2$

[Same as the spectrum of gaussian white noise into $G(t)$]

Note $B_X(\omega_1, \omega_2) = \lambda \; \hat{G}(\omega_1) \hat{G}(\omega_2) \overline{\hat{G}(\omega_1 + \omega_2)}$.

3. Work with $F(X,\omega) = F(X,\omega,T_0,T)$, etc.

$$F(G,\omega) = F(X,\omega) + \tilde{G}(\omega) \cdot \alpha \cdot F(Y,\omega)$$

$$F(H,\omega) = F(Z,\omega) + \tilde{H}(\omega) \cdot \beta \cdot F(Y,\omega)$$

$$\frac{1}{T}\langle |F(G,\omega)|^2 \rangle = \frac{1}{T}\left(\langle |F(X,\omega)|^2 \rangle + \alpha^2 |\tilde{G}(\omega)|^2 \langle |F(Y,\omega)|^2 \rangle\right)$$

$\langle F(X,\omega) F(Y,\omega) \rangle = 0$ by assumed independence

So $P_G(\omega) = P_X(\omega) + \alpha^2 |\tilde{G}(\omega)|^2 P_Y(\omega)$

$$P_H(\omega) = P_Z(\omega) + \beta^2 |\tilde{G}(\omega)|^2 P_Y(\omega)$$

[p. 10.]

$$F(G+H,\omega) = F(X,\omega) + F(Z,\omega) + (\tilde{G}(\omega)\alpha + \tilde{H}(\omega)\beta) F(Y,\omega)$$

$$P_{G+H}(\omega) = P_X(\omega) + P_Z(\omega) + |\alpha \tilde{G}(\omega) + \beta \tilde{H}(\omega)|^2 P_Y(\omega)$$

[again, XY, XZ, YZ terms have $\langle \; \rangle = 0$]

(3)

Power Spectra 1-14 Ans

3, ctd.

$$P_{G+iH}(\omega) = P_{x}(\omega) + P_{z}(\omega) + |\alpha\hat{G}(\omega) + i\beta\hat{H}(\omega)|^2 P_y(\omega)$$

$$\mathrm{Re}\, C_{GH}(\omega) = \tfrac{1}{2}\left(|\alpha\hat{G}(\omega) + \beta\hat{H}(\omega)|^2 - \alpha^2|\hat{G}(\omega)|^2 - \beta^2|\hat{H}(\omega)|^2\right)P_y(\omega)$$

$$= \tfrac{1}{2}\alpha\beta\left(\hat{G}(\omega)\overline{\hat{H}(\omega)} + \overline{\hat{G}(\omega)}H(\omega)\right)P_y(\omega)$$

$$\mathrm{Im}\, C_{GH}(\omega) = \tfrac{1}{2}\left(|\alpha\hat{G}(\omega) + i\beta\hat{H}(\omega)|^2 - \alpha^2|\hat{G}(\omega)|^2 - \beta^2|\hat{H}(\omega)|^2\right)P_y(\omega)$$

$$= \tfrac{1}{2}\alpha\beta\left(-i\,\hat{G}(\omega)\overline{\hat{H}(\omega)} + i\,\overline{\hat{G}(\omega)}\hat{H}(\omega)\right)P_y(\omega)$$

So $C_{GH}(\omega) = \alpha\beta\,\hat{G}(\omega)\overline{\hat{H}(\omega)}\,P_y(\omega)$

For $x = z = 0$:

$$\frac{C_{GH}(\omega)}{\sqrt{P_G(\omega)P_H(\omega)}} = \hat{G}(\omega)\overline{\hat{H}(\omega)}$$