Groups, Fields, and Vector Spaces

Homework #3 (2008) for pages 9-16 of notes-answers

Q1: Coordinate-dependent isomorphisms of vector spaces.

Given:
Vector space V (with elements v, …) and a basis set $\{e_1, e_2, \ldots, e_M\}$
Vector space W (with elements w, …) and a basis set $\{f_1, f_2, \ldots, f_N\}$
We'll construct two vector spaces of dimension $M \times N$, $V \otimes W$ *and* $Hom(V,W)$. *We will then see what happens to the coordinates in these vector spaces when we change basis sets in V and W. to new basis sets,* $\{e'_1, e'_2, \ldots, e'_M\}$ *for V and* $\{f'_1, f'_2, \ldots, f'_N\}$ *for W. The new and old basis sets are related by* $e_i = \sum_{k=1}^{M} A_{ik} e'_k$ *and* $f_j = \sum_{l=1}^{N} B_{jl} f'_l$.

A. As discussed in class (notes pg 16), the vector space $V \otimes W$ *has a basis set* $\{e_1 \otimes f_1, e_1 \otimes f_2, \ldots, e_2 \otimes f_1, \ldots, e_M \otimes f_N\}$, *i.e., any element z of* $V \otimes W$ *can be written in coordinates as* $z = \sum_{i=1, j=1}^{M,N} z_{ij} \left(e_i \otimes f_j\right)$, *for some* $M \times N$ *array of scalars* z_{ij}.

The exercise is to express $z = \sum_{i=1, j=1}^{M,N} z_{ij} \left(e_i \otimes f_j\right)$ *in terms of the new basis set for* $V \otimes W$, *namely as a sum* $z = \sum_{k=1, l=1}^{M,N} z'_{kl} \left(e'_k \otimes f'_l\right)$. *That is, find* z'_{kl} *in terms of* z_{ij}.

From $e_i = \sum_{k=1}^{M} A_{ik} e'_k$, $f_j = \sum_{l=1}^{N} B_{jl} f'_l$, and the linearity of the tensor product, we find $e_i \otimes f_j = \left(\sum_{k=1}^{M} A_{ik} e'_k\right) \otimes \left(\sum_{l=1}^{N} B_{jl} f'_l\right) = \sum_{k=1, l=1}^{M,N} A_{ik} B_{jl} \left(e'_k \otimes f'_l\right)$. So, $z = \sum_{i=1, j=1}^{M,N} z_{ij} \left(e_i \otimes f_j\right)$ implies that $z = \sum_{i=1, j=1}^{M,N} z_{ij} \left(\sum_{k=1, l=1}^{M,N} A_{ik} B_{jl} \left(e'_k \otimes f'_l\right)\right) = \sum_{k=1, l=1}^{M,N} \sum_{i=1, j=1}^{M,N} z_{ij} A_{ik} B_{jl} \left(e'_k \otimes f'_l\right)$. Thus, z'_{kl}, which is the coefficient of $e'_k \otimes f'_l$ in z, is

$$z'_{kl} = \sum_{i=1}^{M} \sum_{j=1}^{N} A_{ik} B_{jl} z_{ij} .$$

B. As discussed in class (notes pg 14), the vector space $Hom(V,W)$ *has a basis set* $\{\psi_{11}, \psi_{12}, \ldots, \psi_{MN}\}$ *where* ψ_{ij} *is the homomorphism for which* $\psi_{ij}(e_i) = f_j$ *and* $\psi_{ij}(e_u) = 0$ *for* $u \neq i$. *With the new basis sets for V and W,* $Hom(V,W)$ *has a basis set* $\{\psi'_{11}, \psi'_{12}, \ldots, \psi'_{MN}\}$, *with* $\psi'_{ij}(e'_i) = f'_j$, *and* $\psi'_{ij}(e'_u) = 0$ *for* $u \neq i$. *In the original basis set, any* φ *in* $Hom(V,W)$ *can be written as* $\varphi = \sum_{i=1, j=1}^{M,N} \varphi_{ij} \psi_{ij}$, *for some* $M \times N$ *array of scalars* φ_{ij}. *The exercise is to express* $\varphi = \sum_{i=1, j=1}^{M,N} \varphi_{ij} \psi_{ij}$ *in terms of the new basis set, namely as a sum* $\varphi = \sum_{k=1, l=1}^{M,N} \varphi'_{kl} \psi'_{kl}$. *That is, find* φ'_{kl} *in terms of* φ_{ij}.

From $\varphi = \sum_{k=1,l=1}^{M,N} \varphi'_{kl}\psi'_{kl}$ and $e_i = \sum_{k=1}^{M} A_{ik}e'_k$, we find

$$\varphi(e_i) = \sum_{k=1,l=1}^{M,N} \varphi'_{kl}\psi'_{kl}(\sum_{u=1}^{M} A_{iu}e'_u) = \sum_{k=1,l=1}^{M,N}\sum_{u=1}^{N} A_{iu}\varphi'_{kl}\psi'_{kl}(e'_u) = \sum_{k=1,l=1}^{M,N} A_{ik}\varphi'_{kl} f'_l .$$

where the final equality uses $\psi'_{kl}(e'_k) = f'_l$ and $\psi'_{kl}(e'_u) = 0$ for $u \neq k$.

Now we need to make use of $f_j = \sum_{l=1}^{N} B_{jl} f'_l$. In the original basis set, $\varphi(e_i) = \varphi_{ij} f_j$. So

$$\varphi(e_i) = \sum_{j=1}^{N} \varphi_{ij} f_j = \sum_{j=1}^{N}\sum_{l=1}^{N} B_{jl}\varphi_{ij} f'_l .$$

Putting together the two equations for $\varphi(e_i)$ yields $\sum_{k=1,l=1}^{M,N} A_{ik}\varphi'_{kl} f'_l = \sum_{l=1}^{N} \varphi_{ij} B_{jl} f'_l$, or,

$$\sum_{l=1}^{N} f'_l \left(\sum_{k=1}^{M} A_{ik}\varphi'_{kl} - \sum_{j=1}^{N} B_{jl}\varphi_{ij} \right) = 0 .$$

Because the f'_l are a basis for *W*, they are linearly independent. Therefore the only way that the above equation can be satisfied is if each coefficient of f'_l is zero. That is,

$\sum_{k=1}^{M} A_{ik}\varphi'_{kl} = \sum_{j=1}^{N} B_{jl}\varphi_{ij}$, for all *l*. Note that this is a system of linear equations in the φ'_{kl}. We can solve it if we know the inverse of the matrix *A*, namely, the quantities A^{-1}_{si} for which $\sum_{i=1}^{M} A^{-1}_{si} A_{ik} = \begin{Bmatrix} 1, s = k \\ 0, s \neq k \end{Bmatrix}$. (Convince yourself that the existence of A^{-1} is guaranteed if both the e_i and the e'_i are a basis.)

Finally, from $\sum_{k=1}^{M} A_{ik}\varphi'_{kl} = \sum_{j=1}^{N} B_{jl}\varphi_{ij}$, we calculate

$\sum_{i=1}^{M}\sum_{k=1}^{M} A^{-1}_{si} A_{ik}\varphi'_{kl} = \sum_{i=1}^{M}\sum_{j=1}^{N} A^{-1}_{si} B_{jl}\varphi_{ij}$ and apply $\sum_{i=1}^{M} A^{-1}_{si} A_{ik} = \begin{Bmatrix} 1, s = k \\ 0, s \neq k \end{Bmatrix}$ to find

$$\boxed{\varphi'_{kl} = \sum_{i=1}^{M}\sum_{j=1}^{N} A^{-1}_{ki} B_{jl}\varphi_{ij} .}$$

The "big-picture" point (compare the circled equations) is that for $Hom(V,W)$, A^{-1} is applied to the *V*-component of the basis, while for $V \otimes W$, *A* is applied to the *V*-component of the basis.. So, a change of basis affects $Hom(V,W)$ and $V \otimes W$ differently.

There are two interesting special cases.
First, take $W = k$, so $Hom(V,W) = V^*$ and $V \otimes W = V$ (convince yourself of this!). This exercise thus shows that V^* and *V* transform differently.

Second, take $W = V$, $f = e$, and $B = A$ (so, also, $f' = e'$) The exercise shows how $Hom(V,V)$ changes when coordinates of V are changed, namely, $\varphi'_{kl} = \sum_{i=1}^{M}\sum_{j=1}^{M} A_{ki}^{-1}\varphi_{ij}A_{jl}$. Or, as standard matrices, $\varphi' = A^{-1}\varphi A$.

Q2: Coordinate-independent (natural) isomorphisms of vector spaces.

A. The dual of the dual. Consider $V^{**} = Hom(V^*,k) = Hom(Hom(V,k),k)$. That is, V^{**} contains elements Φ that are linear mappings from V^* to k. In other words, for two elements φ_1 and φ_2 of V^*, $\Phi(a\varphi_1 + b\varphi_2) = a\Phi(\varphi_1) + b\Phi(\varphi_2)$, where addition here is interpreted in V^*.

Construct a homomorphism M from V to V^{**}. That is, for any element w in V, construct an element $\Phi_w = M(w)$ in V^{**}. To do this, you will have to
(i) come up with a rule for how Φ_w acts on elements φ of V^*,
(ii) show that Φ_w is linear on V^*, namely, that $\Phi_w(a\varphi_1 + b\varphi_2) = a\Phi_w(\varphi_1) + b\Phi_w(\varphi_2)$,
(iii) show that the map M from w to Φ_w is linear on V, namely, that $M(qw_1 + rw_2) = qM(w_1) + rM(w_2)$. (Addition on the left is interpreted in V; addition on the right is interpreted in V^{**}. Equivalently, $\Phi_{qw_1+rw_2} = q\Phi_{w_1} + r\Phi_{w_2}$.

(i) Define $\Phi_w(\varphi) = \varphi(w)$. The right-hand side exploits the fact that since φ is in V^*, it is a linear map on elements of *V*.
(ii) As follows:
$\Phi_w(a\varphi_1 + b\varphi_2) = (a\varphi_1 + b\varphi_2)(w)$ (because of how Φ_w is defined, right-hand-side ops are in V^*)
$(a\varphi_1 + b\varphi_2)(w) = a\varphi_1(w) + b\varphi_2(w)$ (because of how addition and scalar multiplication are defined in V^*)
$a\varphi_1(w) + b\varphi_2(w) = a\Phi_w(\varphi_1) + b\Phi_w(\varphi_2)$ (because of how Φ_w is defined; right-hand-side ops are in *k*)
(iii) To show $\Phi_{qw_1+rw_2} = q\Phi_{w_1} + r\Phi_{w_2}$, which is a statement about V^{**}, we must show that for all φ is in V^*, that $\Phi_{qw_1+rw_2}(\varphi) = q\Phi_{w_1}(\varphi) + r\Phi_{w_2}(\varphi)$.
$\Phi_{qw_1+rw_2}(\varphi) = \varphi(qw_1 + rw_2) = q\varphi(w_1) + r\varphi(w_2) = q\Phi_{w_1}(\varphi) + r\Phi_{w_2}(\varphi)$.
In the above, first and third equalities are the definition of Φ_w; second equality is because φ is a homomorphism.

Comment. This means that every element of *V* can be regarded as an element of V^{**}, and this correspondence does not depend on coordinates.

B. Dual homomorphisms. Consider elements Ψ in $Hom(V,W)$. Construct a homomorphism M from $Hom(V,W)$ to $Hom(W^*,V^*)$. That is, given a homomorphism Ψ from V to W, construct a homomorphism $\Psi^* = M(\Psi)$ from W^* to V^*.

Say Ψ is in $Hom(V,W)$. Say ξ is in W^* (so $\xi(w)$ is an element of *k*). Ψ^* has to map ξ to an element of V^*, i.e., $\Psi^*(\xi)$ needs to be defined by how it maps vectors *v* of *V* to field elements. We therefore define $\left(\Psi^*(\xi)\right)(v) = \xi(\Psi(v))$. (Note that since Ψ is in $Hom(V,W)$, then $\Psi(v)$ is an element of *W*, so ξ can act on it to yield a field element.). Properties (ii) and (iii) are straightforward, and shown in a manner analogous to part A.

Comment. Iterating this argument, one can construct $\Psi^{**} = (\Psi^*)^*$, which is a homomorphism from $Hom(V,W)$ to $Hom(V^{**},W^{**})$. In Part A, we saw that every element of *V* can be regarded as an element of V^{**} (and similarly for *W*). Given this identification, one can readily show that $\Psi^{**} = \Psi$.

C. Find a coordinate-free correspondence between $(V \otimes W)^*$ *and* $Hom(V,W^*)$.
Say *B* is an element of $(V \otimes W)^*$. The means that $B(v \otimes w)$ is an element of the field *k*, and this expression is linear in *v* and *w*.

We need to find an element U_B of $Hom(V,W^*)$ that we can naturally associate with B That is, U_B must be a linear map from vectors *v* to elements in the dual of *W*. To define $U_B(v)$ in the dual of *W*, we must define how it carries out a linear map from elements *w* in *W* to the field *k*. So we take $\left(U_B(v)\right)(w) = B(v \otimes w)$.

D. Find a coordinate free-correspondence between $V \otimes W$ *and* $Hom(V^*,W)$.
Say $v \otimes w$ is in $V \otimes W$. We need to find a linear map from $v \otimes w$ to an element $\Phi = Z(v \otimes w)$ in $Hom(V^*,W)$. To define Φ, we need to show how it maps any φ in V^* to elements of *W*. We therefore define $\Phi = Z(v \otimes w)$ as $\left(Z(v \otimes w)\right)(\varphi) = \varphi(v)w$, which makes use of the fact that since φ is in V^*, it maps vectors *v* to scalars.