Groups, Fields, and Vector Spaces

Homework #2 (2008) for pages 4-9 of notes

Q1: Automorphisms. Let *G* = the group of real 2x2 matrices , nonzero determinant, under multiplication.

A. Consider the mapping *T*, defined by $T(M) = M^T$, where M^T is the transpose of *M* (recall: the transpose exchanges rows and columns.) Is *T* an automorphism? What is T^2? Is it automorphism?

B. Consider the mapping *V*, defined by $V(M) = M^{-1}$, where M^{-1} is the matrix inverse of *M*. Is *V* an automorphism? What is V^2? Is it an automorphism?

C. Consider $\psi = TV$. Is ψ an automorphism? Is ψ^2 an automorphism?

D. An "inner" automorphism is an automorphism which can be written as $\varphi_A(M) = AMA^{-1}$, for some *A*. Which of the above automorphisms are "inner"? Hint: recall a basic property of the determinant: $\det(XY) = \det(X)\det(Y)$. (That is, det is a homomorphism from *G* onto the reals, under multiplication.) Calculate $\det(\varphi_A(M))$. Calculate $\det(\psi(M))$.

Q2: Centers. The "center" of a group *G* is the subset of elements α of *G* for $\alpha g = g\alpha$, for all group elements *g*. (For example, the center of a commutative group is the whole group.)

A. Show that the center of a group is a subgroup.

B. Show that the center is the kernel of the map from *G* into the inner automorphism group of *G*. That is, show that if α is in the center of *G*, then φ_α is the identity map on *G*, and conversely, that if φ_α is the identity map on *G*, then α is in the center of *G*.

C. Find the center of the group of 2x2 matrices in Q1.

Q3. Finite fields.
Display the addition and multiplication tables for a finite field *k* with 4 elements.

Hint: Recall that the additive structure of *k* must be a group of size 4. There are two different ones: $\mathbb{Z}_4$ (the cyclic group of size 4), and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, the direct sum of two groups of size 2. Show that the additive group cannot be $\mathbb{Z}_4$, by the following approach. From $1+1=2$, use the distributive law to show $2 \times 2 = 0$, which cannot happen in a field – since this means that 2 has no multiplicative inverse. Then you only need to find a self-consistent multiplication table, to go along with the additive structure of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Q4. (Bonus): How large is the automorphism group of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$? How large is the automorphism group of $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$? Are they commutative?