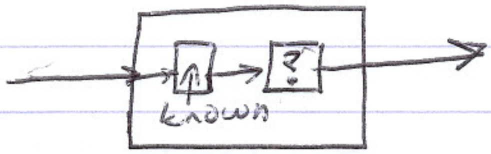
① LINEAR SYSTEMS THEORY

GOAL: provide a concise way of characterizing

$$s(t) \longrightarrow \boxed{F} \longrightarrow r(t)$$

so that one can (a) describe it
(b) test models for it
(c) make guesses as to what's inside
(d) make use of partial knowledge



Examples electric circuit
physiologic
VOR - rotate head, observe eye movement
volume homeostasis - apply volume load, urine output adjust
membrane - apply current measure voltage
light to voltage
neural activity to blood flow

For general F, this is intractable.
So we make 2 assumptions:
translation-invariance
linearity
(+ causality) (Post-processing filters need not be causal.)

②

Translation-invariance: if $[F(s)](t) = r(t)$

and

$$s' = D_\tau(s) \quad (\text{i.e. } s'(t) = s(t+\tau))$$

then

$$[F(s')](t) = r(t+\tau)$$

i.e., $F D_\tau = D_\tau F$

Linearity: $[F(s_1+s_2)](t) = [F(s_1)](t) + [F(s_2)](t)$

and

$$[F(\alpha s)](t) = \alpha [F(s)](t)$$

Linearity is always an approximation.
The theory does extend to non linear transformations, in part.

Linearity allows us to consider F as an element of $\mathrm{Hom}(V,V)$ where V is the V.S. of functions of time.

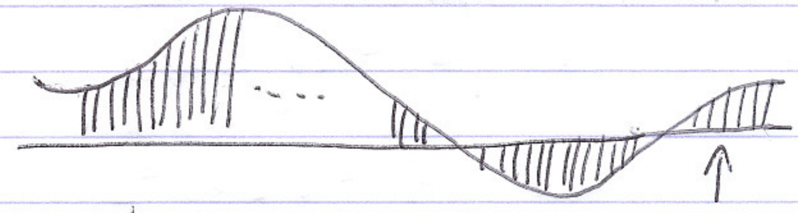
But if F is a physical system, then how to interpret $[F(s)](t)$ for s complex-valued?

Ans: linearity. $s(t) = a(t) + i b(t)$

so $F(s)(t) = [F(a)](t) + i[F(b)](t)$

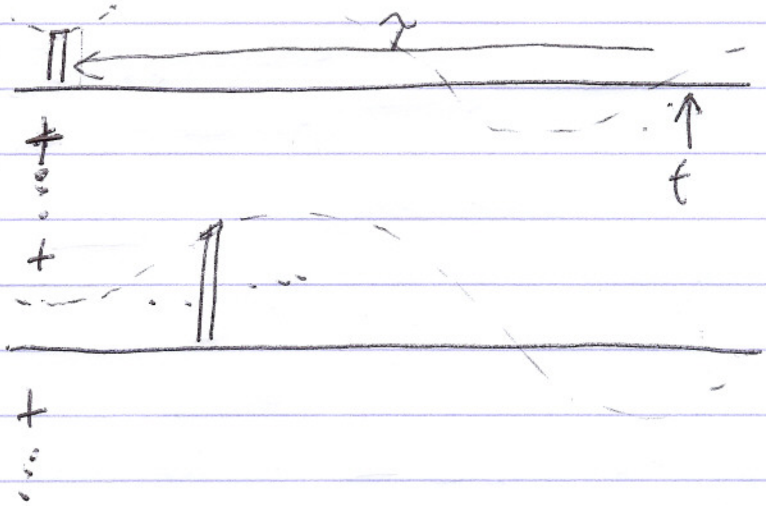
③

Linearity gives us an intuitive way of characterizing F.

s(t) = 

Parcel it into small intervals

Linearity means that we can consider each interval independently.

s(t) = 

Response at time t = sum of contributions of s at all times in past, $t - \tau$.

Say a unit pulse at time 0 leads to a response $f(\tau)$. Equivalently, a unit pulse at time $t-\tau$ leads to a response $f(\tau)$ at time t.

$$r(t) = \int_{\tau=0}^{\infty} f(\tau)\, s(t-\tau)\, d\tau.$$

f is the "impulse response"

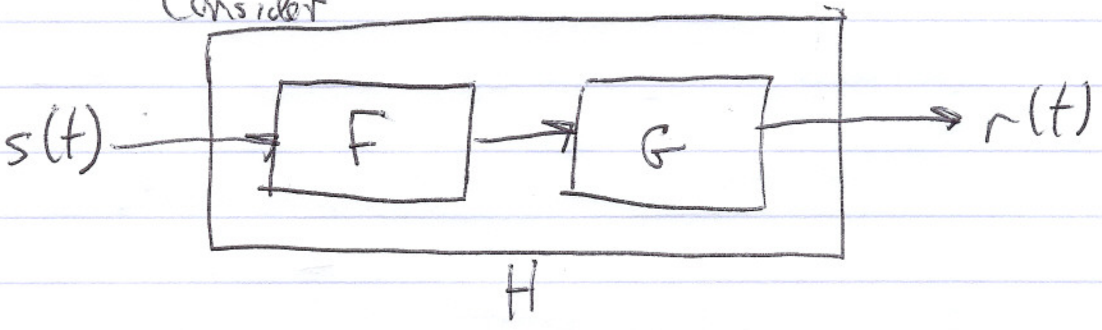
4

What's the problem with this description?

(A) Estimating $f(\tau)$. You need to work where linearity is most likely to break down.

(B). Combining systems. Parallel is easy.

Consider



If you know $f(\tau)$ & $g(\tau)$, what is $h(\tau)$?

say $F(s) = q$, $G(q) = r$.

$$r(t) = \int g(\tau)\, q(t-\tau)\, d\tau$$

$$q(t') = \int f(\tau')\, s(t-\tau')\, d\tau'$$

so $$r(t) = \iint g(\tau)\, f(\tau')\, s(t-\tau-\tau')\, d\tau' d\tau$$

$$u = \tau + \tau'$$

$$= \iint g(\tau)\, f(u-\tau)\, s(t-u)\, d\tau\, du$$

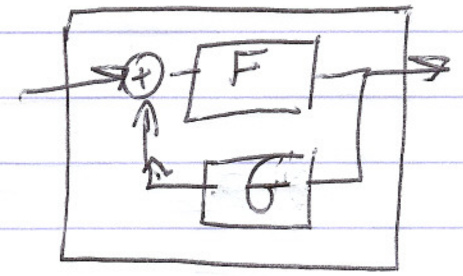
$$= \int h(u)\, s(t-u)\, du \quad \text{for } h(u) = \int g(\tau) f(u-\tau)\, d\tau.$$

So, for systems in series,

$$h(u) = \int g(\tau) f(u-\tau)\, d\tau \quad \text{, or } h = f * g$$

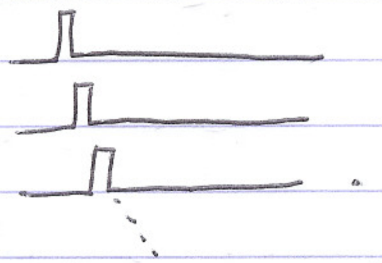
It's not so easy to intuit what this means, or, say, given h and f, to solve for g.

And consider

$H =$ 

or more complex networks

What we've done above is to describe F in terms of the basis set of time functions



These are the delta-functions $\delta(t-\tau)$

D_τ <u>permutes</u> them. But we know that there is another basis set for which D_τ <u>multiplies</u> them, by $e^{i\omega\tau}$.

So now let's change bases.
In the new basis set, i.e., $e^{i\omega t}$, F must act as multiplication.
i.e., for $s(t) = e^{i\omega t}$,
then $[F(s)](t) =$ a multiple of $e^{i\omega t}$.

⑥

Let's find this multiplier call it $\hat{f}(\omega)$, the "transfer function" of F.

If $s(t) = e^{i\omega t}$ then

$$[F(s)](t) = \int f(\tau)\, s(t-\tau)\, d\tau$$

$$= \int f(\tau)\, e^{i\omega(t-\tau)}\, d\tau$$

$$= e^{i\omega t} \int f(\tau) e^{-i\omega\tau}\, d\tau$$

So, $$\boxed{\hat{f}(\omega) = \int e^{-i\omega\tau} f(\tau)\, d\tau.}$$

Interpret $\hat{f}(\omega)$ in terms of real signals. Say $\hat{f}(\omega) = |f(\omega)| e^{i\phi(\omega)}$.

$$s(t) = e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

$$r(t) = |f(\omega)|\, e^{i\phi(\omega)}\, s(t)$$

$$= |f(\omega)| \big(\cos\phi(\omega) + i\sin\phi(\omega)\big)\big(\cos\omega t + i\sin(\omega t)\big)$$

$$= |f(\omega)| \Big([\cos\phi(\omega)\cos\omega t) - \sin\phi(\omega)\sin(\omega t)] + i[\sin\phi(\omega)\cos\omega t + \cos\phi(\omega)\sin(\omega t)] \Big)$$

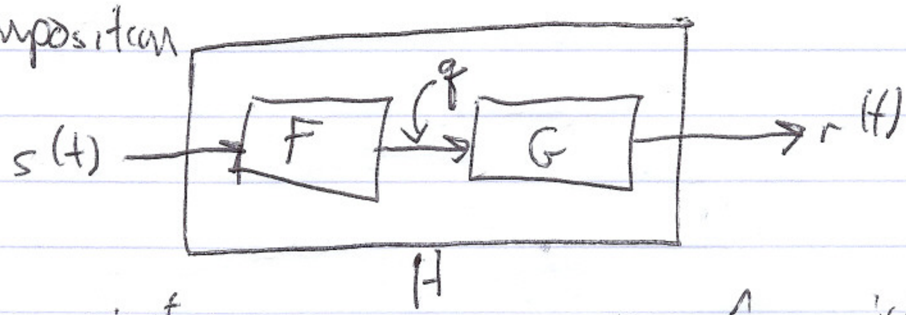
$$= \underset{\substack{\uparrow \\ \text{Amplitude}}}{|f(\omega)|} \Big[\cos(\omega t + \underset{\substack{\uparrow \\ \text{phase shift.}}}{\phi(\omega)}) + i\sin(\omega t + \phi(\omega))\Big]$$

7

Does the transfer function help with the "problems" on p. 4?

(A) measurement. Use ~~$\propto$~~ $\cos(\omega t)$.

(B) composition



Put in $e^{i\omega t} = s(t)$. Then $q(t) = \hat{f}(\omega)\, e^{i\omega t}$.

Next, since G is linear, $r(t) = \hat{g}(\omega)\left[\hat{f}(\omega)\, e^{i\omega t}\right]$.

$$= \hat{g}(\omega)\,\hat{f}(\omega)\, e^{i\omega t}$$

So $\hat{h}(\omega) = \hat{f}(\omega)\,\hat{g}(\omega)$. "The Convolution Thm".

What if we want to calculate how F responds to arbitrary s, but only have $\hat{f}(\omega)$? I.e., can we find $f(t)$ from $\hat{f}(\omega)$?

We need to write $\delta(t) = \int e^{i\omega t} \cdot \hat{\delta}(\omega)\, d\omega$

Since, then we could calculate $f(t) = [F(\delta)](t)$ from its responses to $e^{i\omega t}$.

(8)

"It turns out" that

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\, d\omega$$

so the response to a δ-function is

$$f(t) = [F(\delta)](t) = \frac{1}{2\pi}\int \hat{f}(\omega)\, e^{i\omega t}\, d\omega.$$

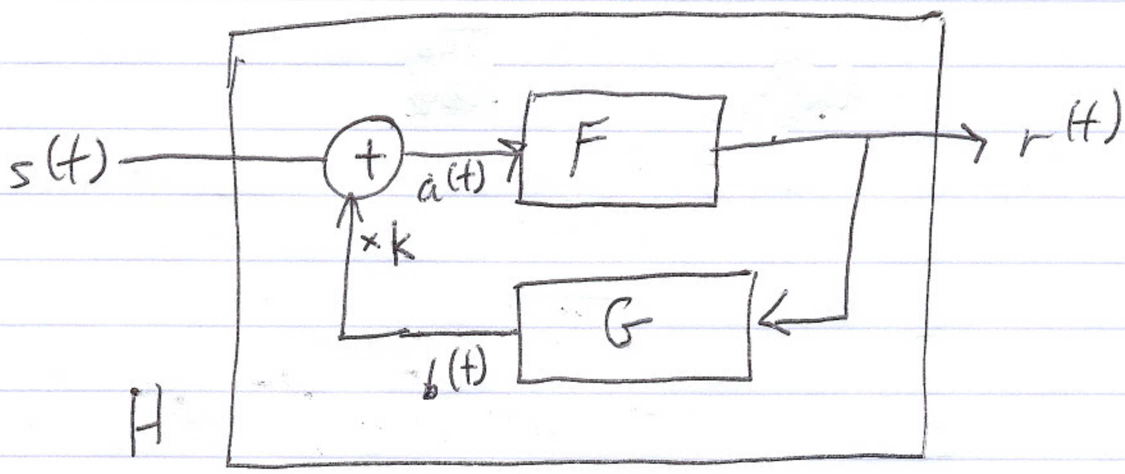
$$\boxed{\begin{aligned} \hat{f}(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt \\ \Updownarrow \qquad & \\ f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\, \hat{f}(\omega)\, d\omega \end{aligned}}$$

Fourier Transform PAIRS

A way to see this - pp 4-5 of 2004 notes FAAP.

⑨

An example of combining elements in a more complex way: feedback



Calculate $\hat{h}(\omega)$. Say $s(t) = e^{i\omega t}$.

$$r(t) = \hat{h}(\omega) e^{i\omega t}.$$

$$b(t) = \hat{g}(\omega)\hat{h}(\omega) e^{i\omega t}$$

$$a(t) = s(t) + k\,b(t) = e^{i\omega t} + k\hat{g}(\omega)\hat{h}(\omega) e^{i\omega t}$$

$$= \underbrace{e^{i\omega t}\left(1 + k\hat{g}(\omega)\hat{h}(\omega)\right)}_{a(t)}$$

$r(t) = [F(a)](t)$, so

$$\hat{h}(\omega) e^{i\omega t} = \hat{f}(\omega) e^{i\omega t}\left(1 + k\hat{g}(\omega)\hat{h}(\omega)\right)$$

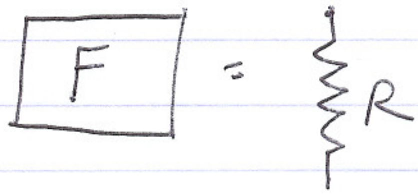
$$\hat{h}(\omega) = \hat{f}(\omega)\left(1 + k\hat{g}(\omega)\hat{h}(\omega)\right)$$

$$\boxed{\hat{h}(\omega) = \frac{\hat{f}(\omega)}{1 - k\hat{f}(\omega)\hat{g}(\omega)}}$$

(10)

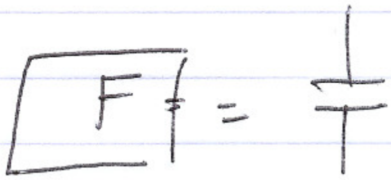
Apply current, measure voltage – in a network of R's + C's.

$$V = IR$$



$$\hat{f}(\omega) = R$$

$$Q = CV$$



$$I = \frac{dQ}{dt} = C\frac{dV}{dt}$$

$$\frac{dV}{dt} = \frac{1}{C} I$$

$I = e^{i\omega t}$, $\quad V = \hat{f}(\omega) e^{i\omega t}$

$$\Rightarrow \frac{dV}{dt} = i\omega e^{i\omega t} \hat{f}(\omega)$$

$$\hat{f}(\omega) i\omega e^{i\omega t} = \frac{1}{C} e^{i\omega t}$$

$$\hat{f}(\omega) = \frac{1}{i\omega C}$$

R's + C's will always lead to algebraic combinations of R + $\frac{1}{i\omega C}$,

i.e., rational expressions in ω.