Nonlinear Systems Theory

Homework #2 (2008) --Answer

Question: How does the Wiener representation depend on input power?

Recall that for a linear-nonlinear-linear sandwich L_1NL_2, where N is characterized by an input-output relationship f, the nth Wiener kernel is given by

$K_n(t_1,t_2,...,t_n) = c_n\int L_1(t_1-\tau)L_1(t_2-\tau)...L_1(t_n-\tau)L_2(\tau)d\tau$ *, where c_n is the nth coefficient in the orthogonal expansion of f with respect to Hermite polynomials based on a Gaussian whose variance P is the variance of the signal that emerges from L_1 That is,*

$c_n = \frac{1}{n!P^n}\int_{-\infty}^{\infty} f(x)h_n(x;P)Gau(x,P)dx$, *where* $Gau(x,P) = \frac{1}{\sqrt{2\pi P}}e^{-x^2/2P}$ *and the Hermite polynomials* $h_n(x;P)$ *have the generating function* $\sum_{n=0}^{\infty}\frac{z^n}{n!}h_n(x;P) = e^{xz-z^2/2P}$. *For example,* $h_0 = 1$, $h_1(x;P) = x$, $h_2(x;P) = x^2 - P$, $h_3(x;P) = x^3 - 3Px$, $h_4(x;P) = x^4 - 6Px^2 + 3P^2$, ….

Given this setup, determine how c_n depends on the power in the input signal. In particular, show that

$$\frac{\partial c_n}{\partial P} = \frac{(n+2)(n+1)}{2}c_{n+2}.$$

Hint: It will be useful to rewrite the Hermite polynomials in a manner that explicitly recognizes how they scale with P, namely, $h_n(x;P) = P^{n/2}H_n(\frac{x}{\sqrt{P}})$, *where*

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}H_n(u) = e^{uz-z^2/2}.$$

Solution (and see important note at the end):

First, the form $h_n(x;P) = P^{n/2}H_n(\frac{x}{\sqrt{P}})$ is implied by (a) noting that as the Gaussian widens by a factor of $\sqrt{P}$, then so do the resulting orthogonal polynomials, and (b) requiring that the leading coefficient (of x^n) is 1.

So $c_n = \frac{1}{n!P^n}\int_{-\infty}^{\infty} f(x)h_n(x;P)Gau(x,P)dx = \frac{1}{n!P^{(n+1)/2}\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)H_n(\frac{x}{\sqrt{P}})e^{-x^2/2P}dx$ and

$$\frac{\partial c_n}{\partial P} = -\left(\frac{n+1}{2}\right)\frac{1}{n!P^{(n+3)/2}\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)H_n(\frac{x}{\sqrt{P}})e^{-x^2/2P}dx$$

$$+\frac{1}{n!P^{(n+1)/2}\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\left(-\frac{x}{2P^{3/2}}H_n'(\frac{x}{\sqrt{P}})+\frac{x^2}{2P^2}H_n(\frac{x}{\sqrt{P}})\right)e^{-x^2/2P}dx\,,$$

where the first term comes from differentiating the factor in front of the integral, and the second term comes from differentiating the integrand. This reduces to

$$\frac{\partial c_n}{\partial P} = \frac{1}{2}\frac{1}{n!P^{(n+3)/2}\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\Big(-(n+1)H_n(u)-uH_n'(u)+u^2H_n(u)\Big)e^{-x^2/2P}dx\,, \qquad (1)$$

where $u = \frac{x}{\sqrt{P}}$. Below we will show that

$$H_{n+2}(u) = -(n+1)H_n(u) - uH_n'(u) + u^2H_n(u)\,. \qquad (2)$$

Using this, eq. (1) becomes

$$\frac{\partial c_n}{\partial P} = \frac{1}{2}\frac{1}{n!P^{(n+3)/2}\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)H_{n+2}(u)e^{-x^2/2P}dx = \frac{1}{2}\frac{1}{n!P^{n/2+1}}\int_{-\infty}^{\infty} f(x)H_{n+2}(u)Gau(x;P)dx\,, (3)$$

or

$$\frac{\partial c_n}{\partial P} = \frac{1}{2}\frac{1}{n!P^{n+2}}\int_{-\infty}^{\infty} f(x)h_{n+2}(x;P)Gau(x;P)dx\,. \qquad (4)$$

Since

$c_{n+2} = \frac{1}{(n+2)!P^{n+2}}\int_{-\infty}^{\infty} f(x)h_{n+2}(x;P)Gau(x,P)dx$, it follows that

$$\frac{\partial c_n}{\partial P} = \frac{1}{2}\frac{(n+2)!}{n!}c_{n+2} = \frac{1}{2}(n+1)(n+2)c_{n+2}\,. \qquad (5)$$

It remains to show eq. (2), which we will do by generating functions. The strategy is to find a generating function for the right-hand side of (2).

Starting with $\sum_{n=0}^{\infty}\frac{z^n}{n!}H_n(u) = G(u,z) = e^{uz-z^2/2}$, we find

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}nH_n(u) = \sum_{n=1}^{\infty}\frac{z^n}{(n-1)!}H_n(u) = z\sum_{n=1}^{\infty}\frac{z^{n-1}}{(n-1)!}H_n(u) = z\frac{\partial}{\partial z}G(u,z) = z(u-z)G(u,z)$$

and

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}H_n{}'(u) = \frac{\partial}{\partial u}G(u,z) = zG(u,z)$$

and so

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}\Big(-(n+1)H_n - uH_n{}'(u) + u^2H_n(u)\Big) = \Big(z(u-z) - 1 - uz + u^2\Big)G(u,z) \qquad (6)$$

$$= \Big((u-z)^2 - 1\Big)G(u,z) = \frac{\partial^2}{\partial z^2}G(u,z)\,.$$

But

$$\frac{\partial^2}{\partial z^2}G(u,z) = \frac{\partial^2}{\partial z^2}\sum_{n=0}^{\infty}\frac{z^n}{n!}H_n(u) = \sum_{n=2}^{\infty}\frac{z^{n-2}}{(n-2)!}H_n(u) = \sum_{n=2}^{\infty}\frac{z^n}{n!}H_{n+2}(u)\,. \qquad (7)$$

Equating coefficients of z^n in (6) and (7) yields (2).

Note

The relationship

$$\frac{\partial c_n}{\partial P} = \frac{1}{2}(n+1)(n+2)c_{n+2} \qquad (8)$$

has implications for nonlinear systems in general, because they are all linear combinations of LN-systems. For an LN-system,

$K_n(t_1, t_2, ..., t_n) = c_n L_1(t_1)L_1(t_2)\cdot ...\cdot L_1(t_n)$. Since the L- factors are independent of the input power,

$$\frac{\partial}{\partial P}K_n(t_1, t_2, ..., t_n) = \frac{(n+1)(n+2)}{2}c_{n+2}L_1(t_1)L_1(t_2)\cdot ...\cdot L_1(t_n)\,. \qquad (9)$$

But $K_{n+2}(t_1, t_2, ..., t_n, t_{n+1}, t_{n+2}) = c_{n+2}L_1(t_1)L_1(t_2)\cdot ...\cdot L_1(t_n)L_1(t_{n+1})L_1(t_{n+2})$, so

$$c_{n+2}L_1(t_1)L_1(t_2)\cdot ...\cdot L_1(t_n)\int_0^{\infty}\left|L_1(t)\right|^2 dt = \int_0^{\infty}K_{n+2}(t_1, t_2, ..., t_n, t, t)dt\,. \qquad (10)$$

In eq. (8), P is the power that gets past L_1. So it is related to the input power P_{input} by

$$P = P_{input}\int_0^{\infty}\left|L_1(t)\right|^2 dt\,. \qquad (11)$$

Combining (9), (10), and (11),

$$\frac{\partial}{\partial P_{input}}K_n(t_1, t_2, ..., t_n) = \frac{P}{P_{input}}\frac{\partial}{\partial P}K_n(t_1, t_2, ..., t_n) = \frac{(n+1)(n+2)}{2}\int_0^{\infty}K_{n+2}(t_1, t_2, ..., t_n, t, t)dt\,. \qquad (12)$$

Since this is true for any LN-system, it is true for any sum of such systems.

Moreover, since the kernels are symmetric in their arguments, Eq. (12) can be interpreted as stating that the derivative of K_n with respect to input power is given by the integral of K_{n+2} over all diagonals in which two of its arguments are equal. There are exactly $\frac{(n+2)(n+1)}{2}$ such diagonals, accounting for the combinatorial factor in eq. (12).