Linear Systems, Black Boxes, and Beyond

Homework #1 (2010-2011)

Q1: Fourier transforms, derivatives, and integrals

Setup is $\hat{s}(\omega)=\int_{-\infty}^{\infty} s(t)e^{-i\omega t}dt$, with $s(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{s}(\omega)e^{+i\omega t}d\omega$.

A. For $q(t)=\frac{d}{dt}s(t)$, find $\hat{q}(\omega)$.

B. For $q_n(t)=\frac{d^n}{dt^n}s(t)$, find $\hat{q}_n(\omega)$.

C. For $z(t)=\int_{-\infty}^{t} s(\tau)d\tau$, find $\hat{z}(\omega)$.

D. Apply C to $s(t)=\delta(t)$ to find a function whose Fourier transform, except possibly at 0, is $\frac{1}{i\omega}$.

Q2: Fourier transforms and moments

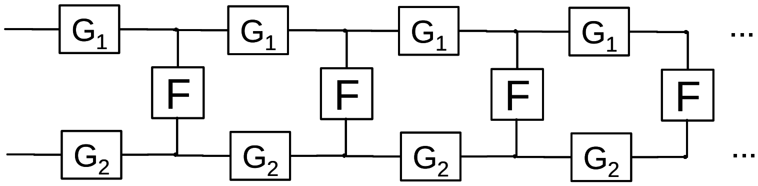
Setup is $\hat{s}(\omega)=\int_{-\infty}^{\infty} s(t)e^{-i\omega t}dt$, with $s(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{s}(\omega)e^{+i\omega t}d\omega$, but now we are thinking of s as a probability distribution.

A. Write the normalization condition $\int_{-\infty}^{\infty} s(t)dt=1$ in terms of $\hat{s}(\omega)$.

B. Write the mean (first moment) $\langle t\rangle=\int_{-\infty}^{\infty} ts(t)dt$ in terms of $s'(\omega)=\frac{d}{d\omega}\hat{s}(\omega)$.

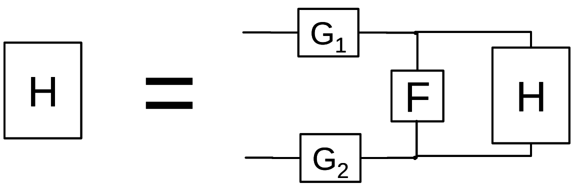
C. Write the variance (second moment) $\left\langle\left(t-\langle t\rangle\right)^2\right\rangle=\langle t^2\rangle-\langle t\rangle^2=\int_{-\infty}^{\infty} t^2 s(t)dt-\left(\int_{-\infty}^{\infty} ts(t)dt\right)^2$ in terms of $s'(\omega)=\frac{d}{d\omega}\hat{s}(\omega)$ and $s''(\omega)=\frac{d^2}{d\omega^2}\hat{s}(\omega)$.

Q3: The half-infinite cable (repeating indefinitely to the right)



This is to be viewed as a network of resistors and capacitors. Calculate the impedance of the system (input applied across terminals at left) in terms of the impedances $F(\omega)$, $G_1(\omega)$, and $G_2(\omega)$ for F, G_1, and G_2.

Hint: Let the composite system be H. Note the following, and then write an equation for $H(\omega)$.



Q4. Boxcar smoothing

Boxcar smoothing refers to convolution with the function $s(t)$, where $s(t)=\begin{cases}\frac{1}{L}, |t|\le L/2 \\ 0, |t|>L/2\end{cases}$. Find its Fourier transform. What does it look like? Is this a good way to smoothe?