Groups, Fields, and Vector Spaces

Homework #1 (2012-2013)

Q1: Group or not a group?

Which of the following are groups? If a group, is it commutative? Finite or infinite? If infinite, is it discrete or continuous? If not a group, where does it fail?

A. The even integers $\{...-6,-4,-2,0,2,4,6...\}$, under addition

B. The set of all rotations of a sphere, under composition

C. The set of all reflections of a sphere, under composition

D. The set of all rotations and reflections of a sphere, under composition

E. The set of all transformations T_p defined by $T_p(x)=x^p$, under composition

F. The set of all reflections and rotations of a rectangle, under composition

Q2. Dihedral groups (one step beyond cyclic groups)

The standard group operation is denoted by juxtaposition.

Consider the following distinct elements: e, a, and r. Assume that they compose in a way that obeys the associative law, that e is the identity, that a is of order 2, and that r is of order $n \geq 2$. (Only $n \geq 3$ is interesting, though.) Suppose further that a and r satisfy $ra = ar^{n-1}$, and that the elements of the set $S=\left\{e,r,r^2,...,r^{n-1},a,ar,ar^2,...,ar^{n-1}\right\}$ are all distinct. Show that this set constitutes a group, of size $2n$. (This is known as the “dihedral group” D_n.)

Q3: Normal subgroups

The standard group operation is denoted by juxtaposition.

Definition: A subgroup H of G is said to be a “normal” subgroup if, for any element g of G and any element h of H, the combination ghg^{-1} is also a member of H.

A. Show that if φ is a homomorphism from G to some other group R, then the kernel of φ is a normal subgroup of G. (In class, we will show that the kernel must be a subgroup, here, assume that it is, and show that it is normal as well.)

B. Show that if H is a normal subgroup and b is any element of G, then the right coset Hb is equal to the left coset, bH.

C. Show that if H is a normal subgroup, then any element of the right coset Hb, composed with any element of the right coset Hc, is a member of the right coset Hbc, with the product bc carried out according to the group operation in G.

D. Consider the mapping from group elements to cosets, $\varphi(b) = Hb$. Show that this constitutes a homomorphism from the group G to the set of cosets, with the group operation on cosets defined by $(Hb) \circ (Hc) = Hbc$.

E. Find the kernel of the homomorphism in D.