Groups, Fields, and Vector Spaces

Homework #3 (2012-2013)

(Q3 refers to first part of notes on "Linear Transformations and Group Representations")

Q1. Intrinsic relationships among dual spaces, etc.

The point of this question is to further illustrate the distinction between vector spaces that have the same abstract structure just because they have the same number of dimensions, and vector spaces for which there is a natural, coordinate-free correspondence.

A. Find an intrinsic relationship (a.k.a. "canonical homomorphism") between *V* and V^{**}. (V^{**} is the dual of V^*, i.e., the space of mappings from elements φ of V^* to the field.) That is, find a linear mapping Φ from elements *v* of *V* to elements $\Phi(v)$ of V^{**}.

B. Find an intrinsic relationship (a.k.a. "canonical homomorphism") between $Hom(V,W)$ and $Hom(W^*,V^*)$. That is, find a linear mapping *Z* from elements φ of $Hom(V,W)$ to elements $Z(\varphi)$ of $Hom(W^*,V^*)$.

C. Find an intrinsic relationship (a.k.a. "canonical isomorphism") between $(V\otimes W)^*$ and $Hom(V,W^*)$. That is, (a) given an element *B* of $(V\otimes W)^*$, find a linear mapping Φ that takes elements *B* of $(V\otimes W)^*$ to elements $\Phi(B)$ of $Hom(V,W^*)$. (b) Given an element ξ of $Hom(V,W^*)$, find a linear mapping Ψ that takes elements ξ of $Hom(V,W^*)$ to elements $\Psi(\xi)$ of $(V\otimes W)^*$. (c) Show that Φ and Ψ are inverses, i.e., $\Psi(\Phi(B))=B$ and $\Phi(\Psi(\xi))=\xi$.

Q2. Parity

A. What is the parity of a cyclic permutation of q elements, i.e., the permutation that puts 2 where 1 was, puts 3 where 2 was, puts 4 where 3 was, …, puts *q* where $q-1$ was, and puts 1 where *q* was?

B. Recall the dihedral group: the symmetry group of a regular *n*-gon, containing rotations by $2\pi k/n$ radians, and reflections. (a) It can also be considered a permutation group, because it permutes the vertices of the *n*-gon. Which group elements correspond to a permutation with an even parity, and which to an odd parity? (b) The dihedral group can also be considered a permutation group in another way, because it acts on the edges of the *n*-gon – i.e., a rotation or a reflection of the dihedral group is a permutation on the edges. In this representation, which group elements correspond to permutations with even parity, and which ones to an odd parity?

Q3. Eigenvectors and Eigenvalues

Setup common to all parts: *A* is in $Hom(V,V)$, where *V* is of dimension *m,* and it has *m* eigenvectors v_j and *m* corresponding eigenvalues λ_j; $Av_j=\lambda_j v_j$. *B* is in $Hom(W,W)$, where

W is of dimension n, and it has n corresponding eigenvectors w_j and n eigenvalues μ_k; $Bw_k = \mu_k w_k$. We assume that all eigenvalues are distinct.

A. We can define a linear transformation $A \oplus B$ in $Hom(V \oplus W, V \oplus W)$ by its action on elements of $V \oplus W$, namely, $(A \oplus B)(v,w) = (Av, Bw)$. What are the eigenvectors and eigenvalues of $A \oplus B$? What is its trace? What is its determinant?

B. We can define a linear transformation $A \otimes B$ in $Hom(V \otimes W, V \otimes W)$ by its action on the elementary tensor products $v \otimes w$ in $V \otimes W$, namely, $(A \otimes B)(v \otimes w) = Av \otimes Bw$ (as in notes). What are the eigenvectors and eigenvalues of $A \otimes B$? What is its trace? What is its determinant?

C. As a special case of part B, we can take $W = V$ and $B = A$. So now $A \otimes A$ has an action in $V \otimes V$, and also in its subspaces $sym(V^{\otimes 2})$ and $anti(V^{\otimes 2})$. Let $A \otimes_{sym} A$ denote the action of $A \otimes A$ in $sym(V^{\otimes 2})$, and, similarly, for $A \otimes_{anti} A$. What are the eigenvectors and eigenvalues of $A \otimes_{sym} A$ and $A \otimes_{anti} A$?

D. What are the eigenvectors and eigenvalues of A^2?