Graph-Theoretic Methods

Homework #1 (2012-2013), Answers

Q1: What is the sum of the eigenvalues of the graph Laplacian?

The sum of the eigenvalues is the trace, i.e., the sum of the diagonal elements. Since the diagonal elements of the graph Laplacian is the degree of each vertex, the trace is the sum of the degrees. Since each edge contributes to the degree of two vertices, the sum of the degrees is twice the number of edges.

Q2: Recall that the vertex incidence matrix Q of a graph, a rectangular matrix with a row for each edge and a column for each vertex that has the property that $L = Q^T Q$ *.* QQ^T *is a matrix that has the same eigenvalues as L. Provide an interpretation for* $QQ^T - 2I$ *.*

Each row of *Q* corresponds to an edge, and has a 1 or a -1 in the columns corresponding to each vertex that is at an endpoint of the edge. An on-diagonal element of QQ^T is therefore equal to 2, since each row contains two nonzero entries. An off-diagonal element of QQ^T is zero unless the two edges contain a common vertex. In that case, the entry is ± 1 (with the sign determined arbitrarily via the original assignment of signs in *Q*). So $QQ^T - 2I$ is the adjacency matrix of the "dual graph" composed of the edges of *G*, with signs assigned (somewhat) arbitrarily.