Linear Systems, Black Boxes, and Beyond

Homework #1 (2012-2013), Answers

Q1: Fourier transforms, derivatives, and integrals

Setup is $\hat{s}(\omega)=\int_{-\infty}^{\infty} s(t)e^{-i\omega t}dt$, *with* $s(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{s}(\omega)e^{+i\omega t}d\omega$.

A. For $q(t)=\frac{d}{dt}s(t)$, *find* $\hat{q}(\omega)$.

Using the "synthesis" integral,

$$q(t)=\frac{d}{dt}s(t)=\frac{d}{dt}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{s}(\omega)e^{+i\omega t}d\omega\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{s}(\omega)\frac{d}{dt}\left(e^{+i\omega t}\right)d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{s}(\omega)\left(i\omega e^{+i\omega t}\right)d\omega .$$

So the coefficient of $e^{i\omega t}$ in $q(t)=\frac{d}{dt}s(t)$ is $\hat{q}(\omega)=i\omega\hat{s}(\omega)$.

B. For $q_n(t)=\frac{d^n}{dt^n}s(t)$, *find* $\hat{q}_n(\omega)$.

Iterating part A: $\hat{q}_n(\omega)=i\omega\hat{q}_{n-1}(\omega)$, so $\hat{q}_n(\omega)=\left(i\omega\right)^n\hat{s}(\omega)$.

C. For $z(t)=\int_{-\infty}^{t}s(\tau)d\tau$, *find* $\hat{z}(\omega)$.

Since $s(t)=\frac{dz}{dt}$, we can use part A: $\hat{s}(\omega)=i\omega\hat{z}(\omega)$, so, except possibly at $\omega=0$, $\hat{z}(\omega)=\frac{\hat{s}(\omega)}{i\omega}$

D. Apply C to $s(t)=\delta(t)$ *to find a function whose Fourier transform, except possibly at 0, is* $\frac{1}{i\omega}$.

Since the Fourier transform of the delta-function is 1 everywhere, the integral of the delta-function, $h(t)=\int_{-\infty}^{t}\delta(\tau)d\tau$ has the required Fourier transform $\frac{1}{i\omega}$. Since the delta-function is an infinitesimally narrow peak with a unit area, the integral evaluates as $h(t)=\begin{cases}1, t>0\\ 0, t<0\end{cases}$. This is the "Heaviside step function." Its value at zero, which is formally undefined, is irrelevant for most purposes.

Q2: Fourier transforms and moments

Setup is $\hat{s}(\omega)=\int_{-\infty}^{\infty} s(t)e^{-i\omega t}dt$, *with* $s(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{s}(\omega)e^{+i\omega t}d\omega$, *but now we are thinking of s as a probability distribution.*

A. Write the normalization condition $\int_{-\infty}^{\infty} s(t)dt=1$ *in terms of* $\hat{s}(\omega)$.

Since $e^{i\omega t}=1$ for $\omega=0$, $\hat{s}(0)=\int_{-\infty}^{\infty} s(t)dt$, so the normalization condition is $\hat{s}(0)=1$.

B. Write the mean (first moment) $\langle t\rangle=\int_{-\infty}^{\infty} ts(t)dt$ *in terms of* $s'(\omega)=\frac{d}{d\omega}\hat{s}(\omega)$.

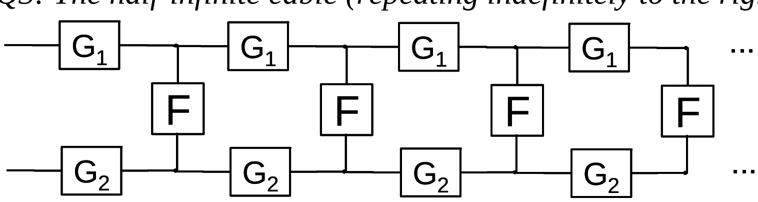
Since $\hat{s}'(\omega)=\int_{-\infty}^{\infty} s(t)\frac{d}{d\omega}e^{-i\omega t}dt=\int_{-\infty}^{\infty} s(t)(-it)e^{-i\omega t}dt$, it follows that $\hat{s}'(0)=\int_{-\infty}^{\infty} s(t)(-it)dt$ and that $\int_{-\infty}^{\infty} ts(t)dt=i\hat{s}'(0)$.

C. Write the variance (second moment) $\left\langle\left(t-\langle t\rangle\right)^2\right\rangle=\langle t^2\rangle-\langle t\rangle^2=\int_{-\infty}^{\infty} t^2s(t)dt-\left(\int_{-\infty}^{\infty} ts(t)dt\right)^2$ *in terms of* $s'(\omega)=\frac{d}{d\omega}\hat{s}(\omega)$ *and* $s''(\omega)=\frac{d^2}{d\omega^2}\hat{s}(\omega)$.

As in part B, $\hat{s}''(\omega)=\int_{-\infty}^{\infty} s(t)\frac{d^2}{d\omega^2}e^{-i\omega t}dt=\int_{-\infty}^{\infty} s(t)(-t^2)e^{-i\omega t}dt$, so $\int_{-\infty}^{\infty} t^2s(t)dt=-\hat{s}''(0)$.

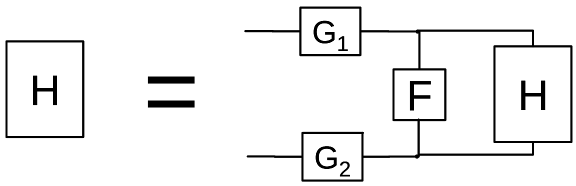
So $\int_{-\infty}^{\infty} t^2s(t)dt-\left(\int_{-\infty}^{\infty} ts(t)dt\right)^2=-\hat{s}''(0)-\left(i\hat{s}'(0)\right)^2=-\hat{s}''(0)+\left(\hat{s}'(0)\right)^2$.

Q3: The half-infinite cable (repeating indefinitely to the right)



This is to be viewed as a network of resistors and capacitors. Calculate the impedance of the system (input applied across terminals at left) in terms of the impedances $F(\omega)$, $G_1(\omega)$, and $G_2(\omega)$ for F, G_1, and G_2.

Hint: Let the composite system be H. Note the following, and then write an equation for $H(\omega)$.



The impedance of the composite system on the left is a series combination of three components: G_1, the parallel combination of *F* and *H*, and G_2. Therefore its impedance is $G_1(\omega)+\frac{F(\omega)H(\omega)}{F(\omega)+H(\omega)}+G_2(\omega)$. Since (as the hint indicates) this is equivalent to the entire half-infinite cable, $H(\omega)=G_1(\omega)+\frac{F(\omega)H(\omega)}{F(\omega)+H(\omega)}+G_2(\omega)$. Solving for $H(\omega)$ yields $H(\omega)^2-G(\omega)H(\omega)-G(\omega)F(\omega)=0$, where $G(\omega)=G_1(\omega)+G_2(\omega)$, or,

$$H(\omega)=\frac{G(\omega)+\sqrt{G(\omega)^2+4F(\omega)G(\omega)}}{2}.$$

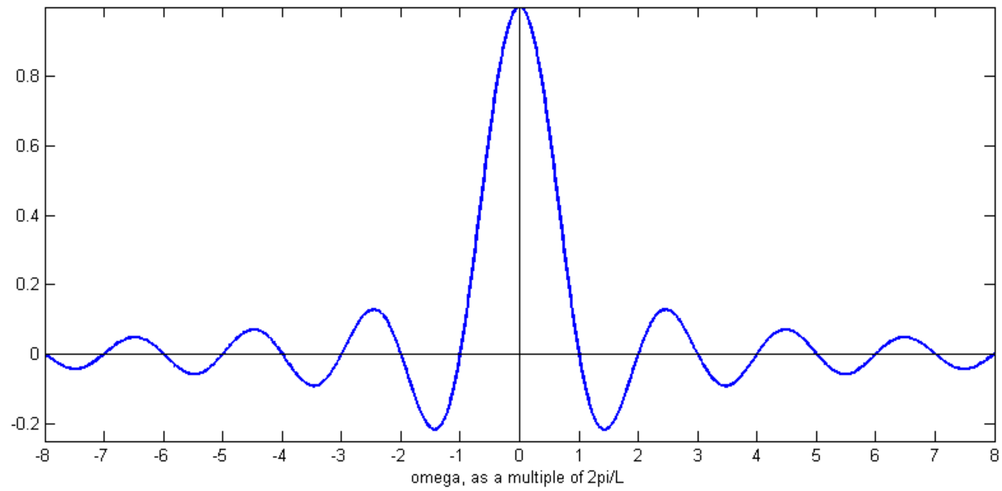
Note concerning the continuum limit: This corresponds to allowing each subunit to represent progressively less and less length. Then *F* has units of impedance/cm (and increases as the segment shortens), and *G* has units of impedance-cm (and decreases as the segment shortens). In this limit, $H(\omega)\approx\sqrt{F(\omega)G(\omega)}$. This enables one to calculate the "cable length" λ, which is the distance required for the transmembrane current to fall by a factor of *e*. To do this, note that total transmembrane current I_{total} is $\int_0^\infty e^{-x/\lambda}dx=\lambda$ times the current per unit length I_{peak} at the injection site, but also, I_{total}/I_{peak} is inversely proportional to the total cable impedance $H(\omega)$, divided by the impedance per unit length, $F(\omega)$. So $\lambda=\frac{H(\omega)}{F(\omega)}=\sqrt{\frac{G(\omega)}{F(\omega)}}$.

Q4. Boxcar smoothing

Boxcar smoothing refers to convolution with the function $s(t)$, where $s(t)=\begin{cases}\frac{1}{L}, |t|\le L/2\\ 0, |t|>L/2\end{cases}$. Find its Fourier transform. What does it look like? Is this a good way to smoothe?

$$\hat{s}(\omega) = \int_{-\infty}^{\infty} s(t) e^{-i\omega t} dt = \frac{1}{L} \int_{-L/2}^{L/2} e^{-i\omega t} dt = \frac{1}{-i\omega L} e^{-i\omega t} \Big|_{-L/2}^{L/2} = \frac{e^{i\omega L/2} - e^{-i\omega L/2}}{i\omega L} = \frac{\sin(\omega L/2)}{(\omega L/2)}.$$

This (the "sinc" function) has a peak of 1 at $\omega = 0$, and descends in an envelope proportional to $1/|\omega|$ away from zero. There are zeros at $\omega = 2\pi k / L$, for $k \neq 0$. The center lobe (at $\omega = 0$) is positive, but the adjacent lobes ($\frac{2\pi}{L} < |\omega| < \frac{4\pi}{L}$) are negative. So one problem with using this as a smoothing function is that it inverts the phase of non-negligible frequency components.

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>> x=[-8:0.01:8];
>> y=sinc(pi*x);
>> plot(x,y)
>> hold on;
>> plot([-8 8],[0 0],'k')
>> plot([0 0],[-0.5 1],'k')
>> set(gca,'YLim',[-0.2 1])
>> set(gca,'YLim',[-0.25 1])
>> xlabel('omega, as a multiple of 2pi/L')
>> set(gca,'XTick',[-8:8])
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