Linear Systems, Black Boxes, and Beyond

Homework #3 (2012-2013), Answers

Q1: Spectra of "gamma" renewal processes. A kth-order gamma process at rate λ *can be constructed by (a) creating a "hidden" Poisson process of rate* $k\lambda$ *, and (b) taking every kth spike.*

A. Write down the Fourier transform of the renewal density for this process.

For the underlying Poisson process of rate $k\lambda$, the renewal density is given by $p_{hidden}(t) = k\lambda e^{-k\lambda t}$, whose Fourier transform is $\hat{p}_{hidden}(\omega) = \int_0^\infty e^{-i\omega t} p_{hidden}(t)dt = \int_0^\infty k\lambda e^{-i\omega t} e^{-k\lambda t} dt = \frac{k\lambda}{-i\omega - k\lambda} e^{-(i\omega + k\lambda)t}\Big|_0^\infty = \frac{1}{1 + i\omega/(k\lambda)}$. Since, by definition, the gamma process looks at every *k*th event, its renewal density is the *k*-fold convolution of this, so

$$\hat{p}_{gamma}(\omega) = \left(\hat{p}_{hidden}(\omega)\right)^k = \left(\frac{1}{1 + i\omega/(k\lambda)}\right)^k.$$

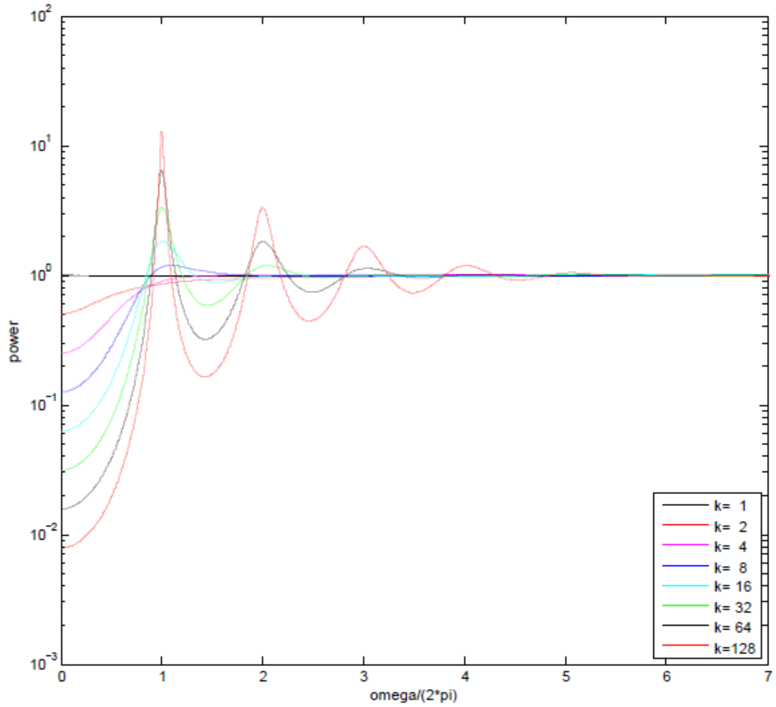
B. Write down the power spectrum for this process.

From the general formula for the power spectrum of a Poisson process,

$$P_{gamma}(\omega) = \lambda \frac{1 - \left|\hat{p}_{gamma}(\omega)\right|^2}{\left|1 - \hat{p}_{gamma}(\omega)\right|^2}$$

C. Graph the power spectrum for $k = \{1, 2, 4, 8, 16, 32, 64, 128\}$.

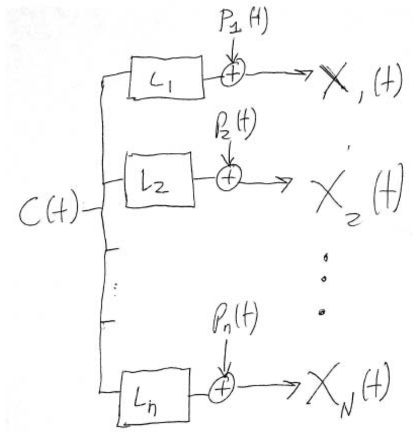
For $\lambda = 1$:



Here's the code:

```
% spec_gammaproc_demo: show the spectrum of gamma processes
%
if ~exist('dw') dw=0.001; end
if ~exist('wmax') wmax=50; end
if ~exist('klist') klist=[1 2 4 8 16 32 64 128]; end
if ~exist('lam') lam=1; end
if ~exist('colors') colors='krmbcg';end
w=[dw:dw:wmax];
figure;
set(gcf,'Position',[100 100 800 700]);
ls=[];
hl=[];
for ik=1:length(klist)
    c=colors(mod(ik-1,length(colors))+1);
    k=klist(ik);
    p=(1./(1+i*w/(k*lam))).^k;
    s=lam*(1-abs(p).^2)./(abs(1-p)).^2;
    hl(ik,1)=semilogy(w/(2*pi),s,c);
    hold on;
    ls=strvcat(ls,sprintf('k=%3.0f',k));
end
xlabel('omega/(2*pi)');
set(gca,'XLim',[0 floor(wmax/(2*pi))]);
ylabel('power');
legend(hl,ls,'Location','SouthEast');
```

Q2: A common noise source feeding multiple observed signals. Consider N observed signals, $X_i(t)$, constructed as follows: There is a common noise source $C(t)$, and $X_i(t)$ results from the addition of this signal, as filtered by L_i, to a "private" noise source $P_i(t)$. The common noise sources and the private noise sources are all assumed to be independent of each other.



A. Calculate the spectra $P_{X_i}(\omega)$ *and the cross-spectra* $P_{X_i X_j}(\omega)$ *in terms of the spectra of the common noise,* $P_C(\omega)$, *the private noises* $P_{P_i}(\omega)$, *and the transfer functions* $\tilde{L}_i(\omega)$. *The spectra and coherences are the elements of the cross-spectral matrix.*

Working from the spectral estimates for X_i:

$$F(X_i,\omega,T,T_0) = \tilde{L}_i(\omega) F(C,\omega,T,T_0) + F(P_i,\omega,T,T_0).$$

Since the private and common noise sources are all independent of each other, for the cross-spectra $(i \neq j)$ we find

$$\left\langle F(X_i,\omega,T,T_0)\overline{F(X_j,\omega,T,T_0)}\right\rangle =$$

$$\left\langle \left(\tilde{L}_i(\omega)F(C,\omega,T,T_0)+F(P_i,\omega,T,T_0)\right)\left(\overline{\tilde{L}_j(\omega)F(C,\omega,T,T_0)+F(P_j,\omega,T,T_0)}\right)\right\rangle =,$$

$$\left\langle \left(\tilde{L}_i(\omega)F(C,\omega,T,T_0)\right)\left(\overline{\tilde{L}_j(\omega)F(C,\omega,T,T_0)}\right)\right\rangle = \tilde{L}_i(\omega)\overline{\tilde{L}_j(\omega)}\left\langle \left|F(C,\omega,T,T_0)\right|^2\right\rangle$$

which imples that

$$P_{X_i X_j}(\omega) = \tilde{L}_i(\omega)\overline{\tilde{L}_j(\omega)} P_C(\omega).$$

For the spectra ($i = j$) we find the cross-term involving the private noise sources does not go away, since it is a product of a private noise term with itself:

$$\left\langle F(X_i,\omega,T,T_0)\overline{F(X_i,\omega,T,T_0)}\right\rangle =$$

$$\left\langle \left(\tilde{L}_i(\omega)F(C,\omega,T,T_0)+F(P_i,\omega,T,T_0)\right)\left(\overline{\tilde{L}_i(\omega)F(C,\omega,T,T_0)+F(P_i,\omega,T,T_0)}\right)\right\rangle =$$

$$\left\langle \left(\tilde{L}_i(\omega)F(C,\omega,T,T_0)\right)\left(\overline{\tilde{L}_j(\omega)F(C,\omega,T,T_0)}\right)\right\rangle + \left\langle F(P_i,\omega,T,T_0)\overline{F(P_i,\omega,T,T_0)}\right\rangle = ,$$

$$\left|\tilde{L}_i(\omega)\right|^2 \left\langle \left|F(C,\omega,T,T_0)\right|^2\right\rangle + \left\langle \left|F(P_i,\omega,T,T_0)\right|^2\right\rangle$$

which implies that

$$P_{X_i}(\omega) = \left|\tilde{L}_i(\omega)\right|^2 P_C(\omega) + P_{P_i}(\omega).$$

B. In the special case that all of the private noises are 0, calculate the global coherence. As in Cimenser et al. (PNAS 2011, http://www.pnas.org/content/108/21/8832.full), the global coherence is the ratio of the first eigenvector of the cross-spectral matrix, to its trace.

Here, the elements of the cross-spectral matrix $M_{ij}(\omega)$ are $P_{X_i X_j}(\omega) = \tilde{L}_i(\omega)\overline{\tilde{L}_j(\omega)}P_C(\omega)$. This matrix is of rank 1, since $M_{ij}(\omega) = A_i(\omega)A_j(\omega)^*$, where $A_i(\omega) = \tilde{L}_i(\omega)\sqrt{P_C(\omega)}$. So there is only one nonzero eigenvalue, namely, $\sum_i A_i(\omega)A_i(\omega)^* = P_C(\omega)\sum_i \left|\tilde{L}_i(\omega)\right|^2$. So the trace (which is the sum of the eigenvalues) is also equal to this quantity, and the global coherence is 1.

C. Calculate the global coherence under the assumption that common noise source $C(t)$ is 0, and all of the private noise sources have the same power spectrum $P_P(\omega)$.

Here, $P_C(\omega) = 0$ so the elements of the cross-spectral matrix are given by $M_{ij}(\omega) = P_{X_i X_j}(\omega) = 0$ if $i \neq j$ and, on the diagonal, $M_{ii}(\omega) = P_{X_i}(\omega) = P_P(\omega)$. So the cross-spectral matrix is $P_P(\omega)$ times the identity matrix. All N eigenvalues are therefore $P_P(\omega)$, so the global coherence is $1/N$.