Linear Transformations and Group Representations

Homework #1 (2012-2013), Answers

Q1: Eigenvectors of some linear operators in matrix form (also see Homework from "Algebraic Overview" (2008-2009))

In each case, find the eigenvalues, the eigenvectors, the dimensions of the eigenspaces, and whether a basis can be chosen from the eigenvectors.

A. $A=\begin{pmatrix}1 & r\\ 0 & 1\end{pmatrix}$.

First, use the determinant to find the eigenvalues. $\det(zI-A)=\det\begin{pmatrix}z-1 & -r\\ 0 & z-1\end{pmatrix}=(z-1)^2$.

$\det(zI-A)=0$ requires $z=1$, so the only eigenvalue of *A* is 1.

Say *V* has basis elements e_1 and e_2, expressed as columns $e_1=\begin{pmatrix}1\\0\end{pmatrix}$ and $e_2=\begin{pmatrix}0\\1\end{pmatrix}$. Then $Ae_1=e_1$ and $Ae_2=re_1+e_2$. So e_1 is an eigenvector of eigenvalue 1. To look for any others: Let $v=ae_1+be_2$. Then $Av=A(ae_1+be_2)=ae_1+b(re_1+e_2)=(a+br)e_1+be_2$.
$Av=v$ implies $ae_1+be_2=(a+br)e_1+be_2$. Since e_1 and e_2 are linearly independent (they form a basis), their coefficients must be equal. For e_1, this requires $a=a+br$, i.e., $b=0$. For e_2, the coefficients are always equal. So the only eigenvalues have $b=0$, i.e., the only eigenvalues are e_1 and its multiples.

So there is one eigenvalue, 1, whose eigenspace has dimension 1, spanned by the eigenvector $e_1=\begin{pmatrix}1\\0\end{pmatrix}$. Since *A* operates in a two-dimensional vector space, the eigenvectors cannot form a basis.

B. $B=\begin{pmatrix}q & r\\ r & q\end{pmatrix}$ *(assume* $q>r>0$*).*

Again, first use the determinant to find the eigenvalues.
$\det(zI-B)=\det\begin{pmatrix}z-q & -r\\ -r & z-q\end{pmatrix}=(z-q)^2-r^2$. $\det(zI-B)=0$ solves for $z=q\pm r$, so these are the eigenvalues of *B*. To find the eigenvectors: As in part *A*, say $e_1=\begin{pmatrix}1\\0\end{pmatrix}$ and $e_2=\begin{pmatrix}0\\1\end{pmatrix}$ and *v* is an eigenvector with $v=ae_1+be_2$. $Be_1=qe_1+re_2$. $Be_2=re_1+qe_2$. So
$Bv=aBe_1+bBe_2=a(qe_1+re_2)+b(re_1+qe_2)=(aq+br)e_1+(ar+bq)e_2$.

Looking for the eigenvector of eigenvalue $q+r$:

$Bv = (q+r)v$ implies $(aq+br)e_1 + (ar+bq)e_2 = (q+r)ae_1 + (q+r)be_2$. Since e_1 and e_2 are linearly independent, equality can only hold if coefficients of e_1 match, and coefficients of e_2 match.
For e_1: $aq+br = aq+ar$; For e_2: $ar+bq = bq+br$. Both solve for $a = b$. So the eigenvectors corresponding to the eigenvalue $q+r$ are multiples of $e_1 + e_2$, i.e., of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For the eigenvectors of eigenvalue $q-r$:

$Bv = (q-r)v$ implies $(aq+br)e_1 + (ar+bq)e_2 = (q-r)ae_1 + (q-r)be_2$. Since e_1 and e_2 are linearly independent, equality can only hold if coefficients of e_1 match, and coefficients of e_2 match.
For e_1: $aq+br = aq-ar$; For e_2: $ar+bq = bq-br$. Both solve for $a = -b$. So the eigenvectors corresponding to the eigenvalue $q-r$ are multiples of $e_1 - e_2$, i.e., of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
So there are two eigenvalues, $q+r$ and $q-r$, each with eigenspace of dimension 1, spanned by $e_1 + e_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $e_1 - e_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. They form a basis.

C. $C = \begin{pmatrix} q & r \\ -r & q \end{pmatrix}$
Again, first use the determinant to find the eigenvalues.
$\det(zI - B) = \det\begin{pmatrix} z-q & -r \\ r & z-q \end{pmatrix} = (z-q)^2 + r^2$. $\det(zI - B) = 0$ solves for $z = q \pm ir$, so these are the eigenvalues of *B*. To find the eigenvectors: As in part *B*, say $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and *v* is an eigenvector with $v = ae_1 + be_2$. $Ce_1 = qe_1 + re_2$. $Ce_2 = -re_1 + qe_2$. So
$Cv = aCe_1 + bCe_2 = a(qe_1 + re_2) + b(-re_1 + qe_2) = (aq - br)e_1 + (ar + bq)e_2$.

Looking for the eigenvector of eigenvalue $q + ir$:
$Cv = (q+ir)v$ implies $(aq-br)e_1 + (ar+bq)e_2 = (q+ir)ae_1 + (q+ir)be_2$. Since e_1 and e_2 are linearly independent, equality can only hold if coefficients of e_1 match, and coefficients of e_2 match.
For e_1: $aq - br = aq + air$; For e_2: $ar + bq = bq + bir$. Both solve for $a = ib$. So the eigenvectors corresponding to the eigenvalue $q+ir$ are multiples of $e_1 + ie_2$, i.e., of $\begin{pmatrix} 1 \\ i \end{pmatrix}$.

Similarly (or, remembering that everything is symmetric with respect to complex conjugation) the eigenvector associated with $z = q - ir = \overline{q + ir}$ must be $\overline{\begin{pmatrix} 1 \\ i \end{pmatrix}} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

Q2: Eigenvectors of some linear operators in a continuous space

V is a vector space of functions of time. In each case, find the eigenvalues and eigenvectors of the indicated operator, and determine whether the operator is time-translation invariant

A. $Lv(t) = tv(t)$.

If $v(t)$ has eigenvalue λ, then $Lv(t) = \lambda v(t)$ means $\lambda v(t) = tv(t)$, which means that either $t = \lambda$ or $v(t) = 0$. This is satisfied by $v(t) = \begin{cases} a, t = \lambda \\ 0, t \neq \lambda \end{cases}$.

L is not time-translation invariant: $\left(D_T L v\right)(t) = \left(D_T (tv)\right)(t)] = (t+T)v(t+T)$ but $\left(LD_T\right)(t) = L\left(D_T(v)\right)(t) = tv(t+T)$.

B. $Rv(t) = v(-t)$.

If $v(t)$ has eigenvalue λ, then $Rv(t) = \lambda v(t)$ means $\lambda v(t) = v(-t)$, and, that $\lambda^2 v(t) = \lambda v(-t) = v(t)$ which means that $\lambda = 1$ or $\lambda = -1$.

If $\lambda = 1$: Any even-symmetric function (satisfying $v(t) = v(-t)$) is an eigenvector.
If $\lambda = -1$: Any odd-symmetric function (satisfying $v(t) = -v(-t)$) is an eigenvector.

Starting with an arbitrary *f*, $(I + R)f = f + Rf$ is always an even-symmetric function, since $R(I + R)f = (R + R^2)f = (R + I)f$.

Similarly, $(I - R)f = f - Rf$ is always an odd-symmetric function, since $R(I - R)f = (R - R^2)f = (R - I)f = -(I - R)f$.

R is not time-translation invariant: $\left(D_T R v\right)(t) = \left(D_T (Rv)\right)(t)] = v(-(t+T)) = v(-t-T)$ but $\left(RD_T\right)(t) = R\left(D_T(v)\right)(t) = v(-t+T)$.

Note that *R* and all the D_T's form a group, since $R^2 = I$ and $D_T R = RD_{-T}$, vaguely like the dihedral group – but here, continuous and open-ended.

C. $Mv(t) = \frac{d}{dt}v(t)$.

If $v(t)$ has eigenvalue λ, then $Mv(t) = \lambda v(t)$ means $\lambda v(t) = \frac{dv}{dt}(t)$, i.e., that *v* satisfies the differential equation $\frac{dv}{dt} = \lambda v$. Solve by separation of variables

$\frac{dv}{v} = \lambda dt$, which implies $d(\log v) = \lambda dt$, which implies $v(t) = a\exp(\lambda t)$ (or by inspection).
Eigenvectors are thus $v(t) = a\exp(\lambda t)$.
M is time-translation invariant: $\left(D_T M v\right)(t) = \frac{d}{dt}v(t+T) = \left(MD_T v\right)(t)$.

Q3: Knowing vector lengths determines the inner product.

V is a Hilbert space, and $\langle v, w\rangle$ is its inner product. Write $\langle v, w\rangle$ in terms of the squared vector lengths $\left\|av+bw\right\|^2 = \langle av+bw, av+bw\rangle$ for selected values of a and b Hint: consider especially $(a,b) = \{(1,1), (1,-1), (1,i), (1,-i)\}$.

First,

$$\left\|av+bw\right\|^2 = \langle av+bw, av+bw\rangle$$
$$= a\overline{a}\langle v,v\rangle + a\overline{b}\langle v,w\rangle + b\overline{a}\langle w,v\rangle + b\overline{b}\langle w,w\rangle .$$
$$= \left|a\right|^2\left\|v\right\|^2 + \left|b\right|^2\left\|w\right\|^2 + a\overline{b}\langle v,w\rangle + b\overline{a}\langle w,v\rangle$$

So

$\left\|v+w\right\|^2 = \left\|v\right\|^2 + \left\|w\right\|^2 + \langle v,w\rangle + \langle w,v\rangle$,
$\left\|v+iw\right\|^2 = \left\|v\right\|^2 + \left\|w\right\|^2 - i\langle v,w\rangle + i\langle w,v\rangle$,
$\left\|v-w\right\|^2 = \left\|v\right\|^2 + \left\|w\right\|^2 - \langle v,w\rangle - \langle w,v\rangle$, and
$\left\|v-iw\right\|^2 = \left\|v\right\|^2 + \left\|w\right\|^2 + i\langle v,w\rangle - i\langle w,v\rangle$.

So $\langle v,w\rangle = \frac{1}{4}\left(\left\|v+w\right\|^2 + i\left\|v+iw\right\|^2 - \left\|v-w\right\|^2 - i\left\|v-iw\right\|^2\right)$.