Linear Transformations and Group Representations

Homework #2 (2012-2013)

Q1: Demonstrating that the pseudoinverse construction yields a projection.

The notes stated that one could construct a projection onto the range of an operator B as $P_B = B(B^*B)^{-1}B^*$, with the fine print that the inverse of B^*B is only computed within the range of B. Show that P_B is a projection, i.e., show that it is self-adjoint and that $P_BP_B = P_B$.

First, show P_B is self-adjoint:

$P_B{}^* = \left(B(B^*B)^{-1}B^*\right)^* = \left(B^*\right)^*\left((B^*B)^{-1}\right)^* B^* = B\left((B^*B)^{-1}\right)^* B^*$ (second equality: adjoint of a product=product of adjoints in reverse order; third equality: $\left(B^*\right)^* = B$.

But also, $\left((B^*B)^{-1}\right)^* = \left((B^*B)^*\right)^{-1} = \left(B^*\left(B^*\right)^*\right)^{-1} = \left(B^*B\right)^{-1}$ (first equality: adjoint of inverse = inverse of adjoint; second equality: adjoint of product = product of adjoints in reverse order)

So $P_B{}^* = B\left((B^*B)^{-1}\right)^* B^* = B(B^*B)^{-1}B^* = P_B$.

Second, show $P_BP_B = P_B$:

$$\begin{aligned}
P_BP_B &= \left(B(B^*B)^{-1}B^*\right)\left(B(B^*B)^{-1}B^*\right)\\
&= B(B^*B)^{-1}B^*B(B^*B)^{-1}B^*\\
&= B(B^*B)^{-1}(B^*B)(B^*B)^{-1}B^*\\
&= B(B^*B)^{-1}B^* = P_B
\end{aligned}$$
. (Just associative law and definition of inverses)

Q2: Another example of a group representation.

Consider the permutations of a set of $n = 3$ abstract elements, $S = \{a,b,c\}$. There are $n! = 6$ permutations of these 3 elements, and they form a group, G, under composition. Let V be the 3-dimensional vector space V of functions on these elements. We can define a unitary representation of G in $Hom(V,V)$ as follows: The unitary transformation U_σ corresponding to the permutation σ is the transformation that takes the function f to $U_\sigma(f)$, where $\left(U_\sigma f\right)(x) = f(\sigma^{-1}(x))$, where $\sigma^{-1}(x)$ denotes the element of S that is moved to x by σ.

A. Verify that this is a representation. That is, show that composition of permutations σ and τ corresponds to composition of the corresponding transformations U_σ and U_τ, $U_\sigma U_\tau = U_{\sigma\tau}$.

$$\left(U_\sigma\left(U_\tau f\right)\right)(x) = \left(U_\tau f\right)(\sigma^{-1}x) = f(\tau^{-1}\sigma^{-1}x) = f\left((\sigma\tau)^{-1}x\right) = \left(U_{\sigma\tau} f\right)(x)$$

Since this is true for all *f* and *x*, it follows that

$U_\sigma U_\tau = U_{\sigma\tau}$.

B. Choose a basis set for V, as follows: $f_a(x) = \begin{cases} 1, & x = a \\ 0, & x \neq a \end{cases}$, *and similarly for* f_b *and* f_c. *So for any f,* $f = f(a) f_a + f(b) f_b + f(c) f_c$, *i.e.* $f = \begin{pmatrix} f(a) \\ f(b) \\ f(c) \end{pmatrix}$. *In this basis, write the matrix form of* U_σ *for* $\sigma = (ab)$ *(*σ *is the permutation that takes a to b and b to a) and* U_τ *for* $\tau = (abc)$ *(*τ *is the permutation that takes a to b, b to c, and c to a).*

For $\sigma = (ab)$:

$\left(U_\sigma f\right)(a) = f(\sigma^{-1}a) = f(b)$

$\left(U_\sigma f\right)(b) = f(\sigma^{-1}b) = f(a)$, so $U_\sigma f = f(b) f_a + f(a) f_b + f(c) f_c$. So in coordinates,

$\left(U_\sigma f\right)(c) = f(\sigma^{-1}c) = f(c)$

$U_\sigma f = \begin{pmatrix} f(b) \\ f(a) \\ f(c) \end{pmatrix}$. And since $f = \begin{pmatrix} f(a) \\ f(b) \\ f(c) \end{pmatrix}$, $U_\sigma f = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} f$.

For $\tau = (abc)$:

$\left(U_\tau f\right)(a) = f(\tau^{-1}a) = f(c)$

$\left(U_\tau f\right)(b) = f(\tau^{-1}b) = f(a)$, so $U_\tau f = f(c) f_a + f(a) f_b + f(b) f_c$. So in coordinates,

$\left(U_\tau f\right)(c) = f(\tau^{-1}c) = f(b)$

$U_\tau f = \begin{pmatrix} f(c) \\ f(a) \\ f(b) \end{pmatrix}$. And since $f = \begin{pmatrix} f(a) \\ f(b) \\ f(c) \end{pmatrix}$, $U_\tau f = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} f$.