Exam, 2012-2013

Do a total of 12 points or any four complete questions. Show your work!

Q1: 5 parts/5 points, no dependencies
Q2: 3 parts/3 points, no dependencies
Q3: 3 parts/3 points, parts B and C depend on A
Q4: 2 parts/2 points, no dependencies
Q5: 2 parts/2 points, part B depends on A
Q6: 2 parts/2 points, part B depends on A

Q1. Semidirect products (5 parts, 1 point for each part)

Here we define the "semi-direct product," a standard way of building larger groups from smaller ones. Let *H* and *K* be groups, with a homomorphism from *K* into the automorphism group of *H*. That is, for each element $k \in K$, there is an automorphism α_k of *H*, and $\alpha_k \alpha_{k'} = \alpha_{kk'}$.

With this setup, we can define an operation on (h,k) pairs: $(h,k) \circ (h',k') = (h\alpha_k(h'), kk')$, where the composition $h\alpha_k(h')$ takes place in *H*, and the composition kk' takes place in *K*.

A. Show that this operation forms a group, known as the "semidirect product of *H* and *K*".

B. Recall the definition of a normal subgroup: A subgroup *N* of *G* is said to be a "normal" subgroup if, for any element *g* of *G* and any element *n* of *N*, the combination gng^{-1} is also a member of *N*. Determine whether the set of elements $S_K = \{(e_H, k)\}$ is a normal subgroup.

C. Determine whether the set of elements $S_H = \{(h, e_K)\}$ is a normal subgroup.

D. Use the above construction to create a continuous non-commutative group, where *H* and *K* are both commutative.

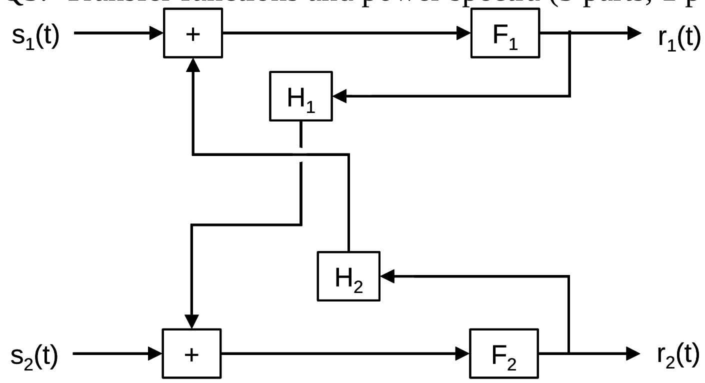
E. Use the above construction to create a discrete non-commutative group that is of size 21.

Q2. Operator exponentials (3 parts, 1 point for each part)

Setup: A is a Hermitian operator; s and t are complex numbers. Define $e^{As} = \sum_{k=0}^{\infty} \frac{1}{k!} s^k A^k$, where A^k indicates (ordinary) repetitive application of the operator A (and $A^0 = I$).

A. Is $e^{A(s+t)} = e^{As} e^{At}$? Why or why not?
B. Is $e^{(A+B)s} = e^{As} e^{Bs}$? Why or why not?
C. What is $\det(e^A)$ in terms of the coefficients of the characteristic equation for A?

Q3. Transfer functions and power spectra (3 parts, 1 point for each part)



The boxes are linear filters, with transfer functions $\tilde{F}_i(\omega)$ and $\tilde{H}_i(\omega)$.

A. Determine the relationships between the Fourier transforms of the inputs $\tilde{s}_i(\omega)$ and the outputs $\tilde{r}_i(\omega)$.

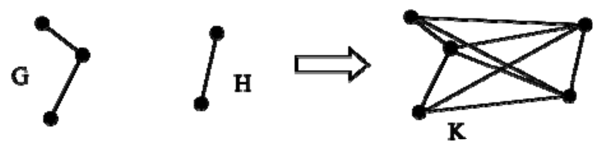
B. Suppose that $s_1(t)$ and $s_2(t)$ are Gaussian noises, whose power spectra are $P_1(\omega)$ and $P_2(\omega)$. Assuming they are independent, calculate the power spectra of $r_1(t)$ and $r_2(t)$, and the cross-spectrum of $r_1(t)$ and $r_2(t)$.

C. As in B, but now assume that $s_1(t)$ and $s_2(t)$ have a nonzero cross-spectrum $X(\omega)$.

Q4. Graph Laplacians of composite graphs (2 parts, 1 point for each part)

A. Say *G* is a connected graph of size n_G whose graph Laplacian L_G has eigenvectors φ_i with eigenvalues λ_i, and *H* is a connected graph of size n_H whose graph Laplacian L_H has eigenvectors ψ_i with eigenvalues μ_i. For definiteness, take φ_1 to be the uniform eigenvector, i.e., the eigenvector composed of all 1's, for which $L_G\varphi_1 = 0$, and, similarly, for ψ_1.

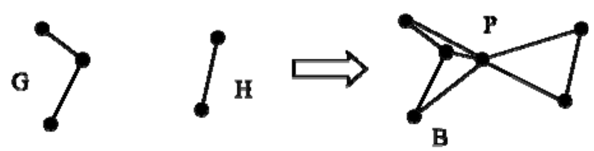
Consider the graph *K* of size $n_G + n_H$ consisting of all the vertices and edges in *G* and *H*, along with an edge from every vertex in *G* to every vertex in *H*. Find the eigenvectors and eigenvalues of the graph Laplacian of *K*.



B. Say *G* is a connected graph of size n_G whose graph Laplacian L_G has eigenvectors φ_i with eigenvalues λ_i, and *H* is a connected graph of size n_H whose graph Laplacian L_H has eigenvectors ψ_i with eigenvalues μ_i. For definiteness, take φ_1 to be the uniform eigenvector, i.e., the eigenvector composed of all 1's, for which $L_G\varphi_1 = 0$, and, similarly, for ψ_1.

Consider the graph *B* of size $n_G + n_H + 1$ consisting of all the vertices and edges in *G* and *H*, along with a new vertex *P*. There are edges from every vertex in *G* to *P*, and from every vertex in *H* to *P* . (If *P* is positioned between *G* and *H*, *B* is a "bowtie").

Find the eigenvectors and eigenvalues of the graph Laplacian of *B.*

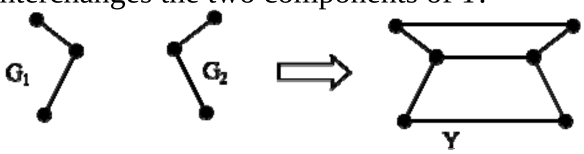


Q5. Graph Laplacians of bilaterally symmetric graphs (2 parts, 1 point for each part)

Say G is a connected graph of size n_G whose graph Laplacian L_G has eigenvectors φ_i with eigenvalues λ_i. For definiteness, take φ_1 to be the uniform eigenvector, i.e., the eigenvector composed of all 1's, for which $L_G\varphi_1 = 0$.

Now form a graph Y that consists of two copies of G, and, between these two copies, the corresponding nodes are connected. (Think of G as being the graph that represents connections within a hemisphere, and Y as being the graph that represents the two hemispheres, with their internal connections and callosal connections between the corresponding areas.)

There's an obvious two-element group $R = \{e, r\}$ that leaves Y invariant: the non-identity element of R interchanges the two components of Y.



A. What does the action of R on Y imply about the eigenvectors of the graph Laplacian L_Y?

B. Determine the eigenvectors and eigenvalues of the graph Laplacian L_Y.

Q6: Community structure (2 parts, 1 point for each part)

Recall that the community structure of a graph consists of an assignment of each vertex *i* to a community c_i that maximizes the "quality function"

$$Q = \sum_{i,j} \left(a_{ij} - p_{ij} \right) \delta(c_i, c_j),$$

where the sum is over all distinct pairs of vertices $\{i, j\}$, $p_{ij} = \dfrac{d_i d_j}{2m}$, where *m* is the total number of edges, and d_i is the degree of the vertex *i*.

A. For a cyclic graph G of size $n \geq 3$, find the community structure.

B. For a linear graph G of size $n \geq 3$, find the community structure.