Groups, Fields, and Vector Spaces

Homework #2 (2014-2015), Questions

Q1: Building larger groups from smaller ones: the general setup

Say *H* and *K* are groups, with identity elements e_H and e_K and group operations $\circ_H$ and $\circ_K$. We define the "direct product" of *H* and *K*, denoted $G = H \times K$, as follows. The elements of *G* are ordered pairs of elements of *H* and *K*, with a typical element denoted $g_i = h_i \times k_i$ with h_i in *H* and k_i in *K*. We define an operation $\circ_G$ in *G* by $(h_1 \times k_1) \circ_G (h_2 \times k_2) = (h_1 \circ_H h_2) \times (k_1 \circ_K k_2)$, i.e., the elements of *G* combine component-wise, according to the operations in their respective groups.

A note on terminology – direct product and direct sum – the terminology is very inconvenient. The "direct product" of two groups is synonymous with the "direct sum", which is denoted $G = H \oplus K$. "Direct sum" (or "direct product") of groups are directly analogous to the "direct sum" or "direct product" construction for vector spaces. But unfortunately the term "direct product" is usually used for groups, and the term "direct sum" is usually used for vector spaces. To avoid confusion with other standard presentations, we will use this unfortunate convention. A further note – for combining an infinite number of groups (or vector spaces), there is a distinction between the direct sum and the direct product– but this is irrelevant to us.

A. Show that the set of g_i form a group, *G*.

B. (optional) For any three groups *H*, *K*, and *M*, construct an isomorphism from $G_{left} = (H \times K) \times M$ into $G_{right} = H \times (K \times M)$. That is, show the results of applying φ to a typical element $g = (h \times k) \times m$ in G_{left} by displaying $\varphi(g)$ in G_{right}, and verify that the group structure of G_{right} is preserved. This result means that we don't care about parentheses in a triple (or larger) direct product, since G_{left} and G_{right} are indistinguishable.

Note also that B shows that the operation $\times$ is associative on the set of groups. Does this operation, along with the set of all groups, form a group?

Q2: Building larger groups from smaller ones: examples

Recall that $\mathbb{Z}_p$ is the group containing the elements $\{0, 1, ..., p-1\}$, with the group operation of addition mod *p* – the "cyclic group" of *p* elements. We denote the group operation by $+$, and use αx as a shorthand for $x + x + \ldots + x$ a total of α times.

A. How many elements are in $\mathbb{Z}_p \times \mathbb{Z}_q$?
B. Is $\mathbb{Z}_3 \times \mathbb{Z}_5$ isomorphic to $\mathbb{Z}_{15}$? Hint: let *h* be a non-identity element of $\mathbb{Z}_3$, and *k* be a non-identity element of $\mathbb{Z}_5$. What is the order of $h \times k$?
C. Is $\mathbb{Z}_3 \times \mathbb{Z}_4$ isomorphic to $\mathbb{Z}_{12}$?

D. Is $\mathbb{Z}_3 \times \mathbb{Z}_6$ isomorphic to $\mathbb{Z}_{18}$?

E. Formulate a hypothesis for when $\mathbb{Z}_p \times \mathbb{Z}_q$ is isomorphic to $\mathbb{Z}_{pq}$, and (optionally) prove it.

Q3: Automorphisms of groups

A. What are the automorphisms of $\mathbb{Z}_5$?

B. What are the automorphisms of $\mathbb{Z}_6$?

C. What are the automorphisms of $\mathbb{Z}_2 \times \mathbb{Z}_2$?

D. (Challenging, optional) What are the automorphisms of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$?