Groups, Fields, and Vector Spaces

Homework #3 (2014-2015), Answers

Q1: Tensor products: concrete examples

Let V and W be two-dimensional vector spaces, with bases $\{v_1, v_2\}$ *and* $\{w_1, w_2\}$. *So* $\{v_i \otimes w_j\}$ *is a basis for* $V \otimes W$. *Say* $x_i \in V$ *has the basis expansion* $x = \alpha_1 v_1 + \alpha_2 v_2$ *and* $y_i \in W$ *has the basis expansion* $y = \beta_1 w_1 + \beta_2 w_2$.

A. Expand $x \otimes y$ *in the basis* $\{v_i \otimes w_j\}$.

$$
\begin{aligned}
& x \otimes y \\
&= (\alpha_1 v_1 + \alpha_2 v_2) \otimes (\beta_1 w_1 + \beta_2 w_2) \\
&= (\alpha_1 v_1 + \alpha_2 v_2) \otimes (\beta_1 w_1) + (\alpha_1 v_1 + \alpha_2 v_2) \otimes (\beta_2 w_2) \\
&= (\alpha_1 v_1) \otimes (\beta_1 w_1) + (\alpha_2 v_2) \otimes (\beta_1 w_1) + (\alpha_1 v_1) \otimes (\beta_2 w_2) + (\alpha_2 v_2) \otimes (\beta_2 w_2) \\
&= \alpha_1 \beta_1 (v_1 \otimes w_1) + \alpha_2 \beta_1 (v_2 \otimes w_1) + \alpha_1 \beta_2 (v_1 \otimes w_2) + \alpha_2 \beta_2 (v_2 \otimes w_2)
\end{aligned}
$$

B. Now say $V = W$*, and we are using the same basis for x and y, so that* $x = \alpha_1 v_1 + \alpha_2 v_2$ *and* $y = \beta_1 v_1 + \beta_2 v_2$. *Expand* $x \otimes y$ *in the basis* $\{v_i \otimes v_j\}$.

Taking $w_i = v_i$ in part A,

$$x \otimes y = \alpha_1 \beta_1 (v_1 \otimes v_1) + \alpha_2 \beta_1 (v_2 \otimes v_1) + \alpha_1 \beta_2 (v_1 \otimes v_2) + \alpha_2 \beta_2 (v_2 \otimes v_2)$$

C. Expand $x \otimes y + y \otimes x$ *in the basis* $\{v_i \otimes v_j\}$.

$$
\begin{aligned}
& (x \otimes y) + (y \otimes x) \\
&= \left(\alpha_1 \beta_1 (v_1 \otimes v_1) + \alpha_2 \beta_1 (v_2 \otimes v_1) + \alpha_1 \beta_2 (v_1 \otimes v_2) + \alpha_2 \beta_2 (v_2 \otimes v_2)\right) \\
&+ \left(\beta_1 \alpha_1 (v_1 \otimes v_1) + \beta_2 \alpha_1 (v_2 \otimes v_1) + \beta_1 \alpha_2 (v_1 \otimes v_2) + \beta_2 \alpha_2 (v_2 \otimes v_2)\right) \\
&= 2\alpha_1 \beta_1 (v_1 \otimes v_1) + (\alpha_2 \beta_1 + \beta_2 \alpha_1)(v_2 \otimes v_1) + (\alpha_1 \beta_2 + \beta_1 \alpha_2)(v_1 \otimes v_2) + 2\alpha_2 \beta_2 (v_2 \otimes v_2)
\end{aligned}
$$

D. Expand $x \otimes y - y \otimes x$ *in the basis* $\{v_i \otimes v_j\}$.

$$
\begin{aligned}
& (x \otimes y) - (y \otimes x) \\
&= \left(\alpha_1 \beta_1 (v_1 \otimes v_1) + \alpha_2 \beta_1 (v_2 \otimes v_1) + \alpha_1 \beta_2 (v_1 \otimes v_2) + \alpha_2 \beta_2 (v_2 \otimes v_2)\right) \\
&- \left(\beta_1 \alpha_1 (v_1 \otimes v_1) + \beta_2 \alpha_1 (v_2 \otimes v_1) + \beta_1 \alpha_2 (v_1 \otimes v_2) + \beta_2 \alpha_2 (v_2 \otimes v_2)\right) \\
&= (\alpha_2 \beta_1 - \beta_2 \alpha_1)(v_2 \otimes v_1) + (\alpha_1 \beta_2 - \beta_1 \alpha_2)(v_1 \otimes v_2) \\
&= (\alpha_2 \beta_1 - \beta_2 \alpha_1)\left((v_2 \otimes v_1) - (v_1 \otimes v_2)\right)
\end{aligned}
$$

Q2: Free vector spaces, direct sums and tensor products

Let V be the free vector space on a set S, namely, the set of all functions v on S, with addition defined pointwise, $(v_1 + v_2)(s) = v_1(s) + v_2(s)$ *, and scalar multiplication defined by* $(\alpha v)(s) = \alpha \cdot (v(s))$ *. Similarly, let W be the free vector space on a set T, namely, the set of all functions w on T, with addition and multiplication defined in an analogous fashion.*

A. Show that the direct-sum vector space $V \oplus W$ *is the same as (i.e., "canonically isomorphic" to) the free vector space on* $S \cup T$ *, the union of the sets S and T . That is, construct an isomorphism between the two spaces, without resorting to choosing a basis.*

This is mostly an exercise in keeping straight what operates on what.

Elements in $V \oplus W$, by definition, consist of pairs of elements in *V* and *W*, added element-by-element, while elements in the free vector space on $S \cup T$ consist of all functions on $S \cup T$. We construct a homomorphism Φ from $V \oplus W$ to the free vector space on $S \cup T$, and a homomorphism Θ from the free vector space on $S \cup T$ back to $V \oplus W$, and show that they are inverses.

Say $z = (v, w)$ is in $V \oplus W$. To find its corresponding element in the free vector space on $S \cup T$, we need to re-interpret it as a function $\Phi(z)$ in $S \cup T$, that is, assign a value to $\left(\Phi(z)\right)(u)$ for every *u* in $S \cup T$. The natural choice is

$\left(\Phi(z)\right)(u) = \begin{cases} v(u) \ if \ u \in S \\ w(u) \ if \ u \in T \end{cases}$. We'll skip some of the details of showing that Φ preserves vector-space structure, but, for example, here are the details for showing $\Phi(z_1 + z_2) = \Phi(z_1) + \Phi(z_2)$:

$z_i = (v_i, w_i)$, then $z_1 + z_2 = (v_1 + v_2, w_1 + w_2)$ and

$\left(\Phi(z_1 + z_2)\right)(u) = \begin{cases} v_1(u) + v_2(u) \ if \ u \in S \\ w_1(u) + w_2(u) \ if \ u \in T \end{cases}$, while $\left(\Phi(z_i)\right)(u) = \begin{cases} v_i(u) \ if \ u \in S \\ w_i(u) \ if \ u \in T \end{cases}$, so

$\left(\Phi(z_1 + z_2)\right)(u) = \left(\Phi(z_1)\right)(u) + \left(\Phi(z_2)\right)(u)$ for all *u*, and hence $\Phi(z_1 + z_2) = \Phi(z_1) + \Phi(z_2)$.

Conversely, say *x* is an element of the free vector space on $S \cup T$. We need to find a $\Theta(x)$ that maps *x* into an ordered pair of elements (v, w) with *v* in the free vector space on *S*, and *w* in the free vector space on *T*, So we take $\Theta(x) = (v, w)$ where $v(s) = x(s)$ and similarly $w(t) = x(t)$, noting that for $x(u)$ is defined for all $u \in S \cup T$, so it is defined both on *S* and *T*.

To show that Θ is a homomorphism, we need to show $\Theta(x_1+x_2)=\Theta(x_1)+\Theta(x_2)$ and $\Theta(\lambda x_1)=\lambda\Theta(x_1)$. We show the first in detail. $\Theta(x_1+x_2)$ is an ordered pair (v,w), where *v* is defined by $v(s)=\Theta(x_1+x_2)(s)=\Theta(x_1)(s)+\Theta(x_2)(s)$, and $w(t)=\Theta(x_1+x_2)(t)=\Theta(x_1)(t)+\Theta(x_2)(t)$. On the other hand, each $\Theta(x_i)$ is an ordered pair (v_i,w_i) with $v_i(s)=\Theta(x_i)(s)$ and $w_i(t)=\Theta(x_i)(t)$. So $v(s)=v_1(s)+v_2(s)$ for all $s\in S$ and $w(t)=w_1(t)+w_2(t)$ for all $t\in T$. So $(v,w)=(v_1,w_1)+(v_2,w_2)$, and $\Theta(x_1+x_2)=(v,w)=(v_1,w_1)+(v_2,w_2)=\Theta(x_1)+\Theta(x_2)$.

Finally, we need to show that the above two constructions are inverses of each other, namely, that $\Theta(\Phi(z))=z$ for $z=(v,w)$ in $V\oplus W$, and that $\Phi(\Theta(x))=x$ for *x* in free vector space on $S\cup T$.

Consider first $\Theta(\Phi(z))$. According to the definition of Θ, this is the element (v,w) of $V\oplus W$ for which $v(s)=(\Phi(z))(s)$ for $s\in S$ and and $w(t)=(\Phi(z))(t)$ for $t\in T$. But $(\Phi(z))(s)=v(s)$ and $(\Phi(z))(t)=w(t)$, according to the definition of Φ. So $\Theta(\Phi(z))=z$. The other way around is equally (un)illuminating.

B. As free vector spaces, recall that V has the "delta-function" basis consisting of the vectors $\delta_{s'}$ defined by $\delta_{s'}(s)=1$ for $s=s'$, and 0 otherwise, and W has the analogous delta-function basis consisting of the vectors $\delta_{t'}$ defined by $\delta_{t'}(t)=1$ for $t=t'$, and 0 otherwise. Display the delta-function basis for $V\oplus W$.

These are the delta-functions on $S\cup T$, namely $\delta_{u'}(u)$ defined by $\delta_{u'}(u)=1$ for $u'=u$ and 0 otherwise. Equivalently, they are extensions of $\delta_{s'}$ and $\delta_{t'}$ to $S\cup T$, giving them a value of 0 beyond the set on which they were originally defined.

C. (optional) Show that the tensor-product vector space $V\otimes W$ is the same as (i.e., "canonically isomorphic" to) the free vector space on $S\times T$, i.e., the set of all ordered pairs (s,t) of elements $s\in S$ and $t\in T$. That is, construct an isomorphism between the two spaces, without resorting to choosing a basis.

As in A, we construct a homomorphism Φ from $V\otimes W$ to the free vector space on $S\times T$, a homomorphism Θ from the free vector space on $S\times T$ back to $V\otimes W$, and show that they are inverses.

First, we construct the mapping Φ for the elementary tensor products $v\otimes w$, i.e., we construct $\Phi(v\otimes w)$, and show that it obeys the tensor-product rules. So we have to specify the value of $v\otimes w$ on a typical element (s,t) in $S\times T$. We define $(\Phi(v\otimes w))(s,t)=v(s)w(t)$, where the multiplication on the right is in the base field. We have to show consistency with the tensor-product rules, namely, that $\Phi(\lambda v\otimes w)=\Phi(v\otimes\lambda w)$ and $\Phi((v_1+v_2)\otimes w)=\Phi(v_1\otimes w)+\Phi(v_2\otimes w)$. We do this by evaluating both

sides on elements $(s,t)\in S\times T$. For the first,
$\Phi(\lambda v\otimes w)(s,t)=\left(\lambda v(s)\right)\left(w(t)\right)=\lambda\left(v(s)\right)\left(w(t)\right)=\left(v(s)\right)\left(\lambda w(t)\right)=\Phi(v\otimes\lambda w)(s,t)$.
For the second,

$$\begin{aligned}\Phi((v_1+v_2)\otimes w)(s,t)&=\left((v_1+v_2)(s)\right)\left(w(t)\right)=\left((v_1(s)+v_2(s)\right)\left(w(t)\right)\\&=\left(v_1(s)\right)\left(w(t)\right)+\left(v_2(s)\right)\left(w(t)\right)=\Phi(v_1\otimes w)(s,t)+\Phi(v_2\otimes w)(s,t)\end{aligned},$$

where we have used the definition of Φ in the first equality, then the definition of addition in the free vector space on *S*, then the distributive law in the base field, and in the final step, the definition of Φ again.

Note that showing consistency with tensor-product rules also shows that Φ is a homomorphism.

Next, we construct a homomorphism Θ from the free vector space on $S\times T$ back to $V\otimes W$. Motivated by the way Φ is defined, we work on elements in the free vector space on $S\times T$ that are the images of elementary tensor products. These are the "separable" elements of the free vector space on $S\times T$, namely, the functions $x\left((s,t)\right)$ for which $x\left((s,t)\right)=v(s)w(t)$. For these functions, we define $\Theta(x)=v\otimes w$. But we need to check that this is a self-consistent definition: since $x(s,t)=\left(v(s)\right)\left(w(t)\right)=\left(\alpha v(s)\right)\left(\frac{1}{\alpha}w(t)\right)$, we need to be sure that applying Θ to the second factorization yields the same result as the first. The second yields $(\alpha v)\otimes\left(\frac{1}{\alpha}w\right)$, and this is guaranteed equal to $v\otimes w$ by the rules for the tensor product in $V\otimes W$: $(\alpha v)\otimes\left(\frac{1}{\alpha}w\right)=\frac{1}{\alpha}\left((\alpha v)\otimes w\right)=\frac{\alpha}{\alpha}\left(v\otimes w\right)=v\otimes w$.

Now, we need to extend this definition to the entire free vector space on $S\times T$, not just the separable elements. We note that the separable elements contain a basis – the delta-function basis, functions $\delta_{v\times w}(s,t)=\delta_v(s)\delta_w(t)$. Each element of the entire free vector space on $S\times T$ can be written uniquely as $x\left((s,t)\right)=\sum_{s',t'}\left(x(s',t')\right)\delta_{s'\times t'}(s,t)$, i.e., $x(s',t')$ is the coefficient of the basis element $\delta_{s'\times t'}$ in the expansion of *x*. So Θ extends to the entire free vector space, by $\Theta(x)=\sum_{s',t'}\left(x(s',t')\right)\delta_{s'}\otimes\delta_{t'}$.

Finally, note that we didn't need to introduce the basis to define the homomorphism Θ, but we did need it to show that it extended to the whole free vector space on $S\times T$. So we need to check that this extended homomorphism coincides with our original definition $\Theta(x)=v\otimes w$ for $x\left((s,t)\right)=v(s)w(t)$.

This holds because if $x\left((s,t)\right)=v(s)w(t)$, then its coefficients in the basis representation also factor: $x(s',t')=v(s')w(t')$. So

$$\begin{aligned}\Theta(x)&=\sum_{s',t'}\left(x(s',t')\right)\delta_{s'}\otimes\delta_{t'}=\sum_{s',t'}\left(v(s')w(t')\right)\delta_{s'}\otimes\delta_{t'}\\&=\sum_{s',t'}\left(v(s')\delta_{s'}\right)\otimes\left(w(t')\delta_{t'}\right)=\left(\sum_{s'}v(s')\delta_{s'}\right)\otimes\left(\sum_{t'}w(t')\delta_{t'}\right)=v\otimes w\end{aligned},$$

where we've used the rules for the tensor product for the third equality, we separated the sum on the fourth equality,and we used the delta-basis representation for *v* and *w* for the final equality.

To show that $\Theta\big(\Phi(z)\big) = z$, we take $z = v \otimes w$ in $V \otimes W$. , Then $\Phi(z)$ is the function on $S \times T$ for which $\Phi(z)\big((s,t)\big) = v(s)w(t)$ (by the definition of Φ). Since this is separable, $\Theta\big(\Phi(z)\big) = v \otimes w$ (from the definition of Θ), as required.

Similarly, to see that $\Phi\big(\Theta(x)\big) = x$, take an *x* for which $x\big((s,t)\big) = v(s)w(t)$. Then $\Theta(x) = v \otimes w$ (from the definition of Θ), and $\Phi\big(\Theta(x)\big)$ is the function on $S \times T$ for which $\Phi\big(\Theta(x)\big)\big((s,t)\big) = v(s)w(t)$ (by the definition of Φ).