Groups, Fields, and Vector Spaces

Homework #3 (2014-2015), Questions

Q1: Tensor products: concrete examples

Let V and W be two-dimensional vector spaces, with bases $\{v_1, v_2\}$ and $\{w_1, w_2\}$. So $\{v_i \otimes w_j\}$ is a basis for $V \otimes W$. Say $x_i \in V$ has the basis expansion $x = \alpha_1 v_1 + \alpha_2 v_2$ and $y_i \in W$ has the basis expansion $y = \beta_1 w_1 + \beta_2 w_2$.

A. Expand $x \otimes y$ in the basis $\{v_i \otimes w_j\}$.

B. Now say $V = W$, and we are using the same basis for x and y, so that $x = \alpha_1 v_1 + \alpha_2 v_2$ and $y = \beta_1 v_1 + \beta_2 v_2$. Expand $x \otimes y$ in the basis $\{v_i \otimes v_j\}$.

C. Expand $x \otimes y + y \otimes x$ in the basis $\{v_i \otimes v_j\}$.

D. Expand $x \otimes y - y \otimes x$ in the basis $\{v_i \otimes v_j\}$.

Q2: Free vector spaces, direct sums and tensor products

Let V be the free vector space on a set S, namely, the set of all functions v on S, with addition defined pointwise, $(v_1 + v_2)(s) = v_1(s) + v_2(s)$, and scalar multiplication defined by $(\alpha v)(s) = \alpha \cdot (v(s))$. Similarly, let W be the free vector space on a set T, namely, the set of all functions w on T, with addition and multiplication defined in an analogous fashion.

A. Show that the direct-sum vector space $V \oplus W$ is the same as (i.e., “canonically isomorphic” to) the free vector space on $S \cup T$, the union of the sets S and T. That is, construct an isomorphism between the two spaces, without resorting to choosing a basis.

B. As free vector spaces, recall that V has the “delta-function” basis consisting of the vectors $\delta_{s'}$ defined by $\delta_{s'}(s) = 1$ for $s = s'$, and 0 otherwise, and W has the analogous delta-function basis consisting of the vectors $\delta_{t'}$ defined by $\delta_{t'}(t) = 1$ for $t = t'$, and 0 otherwise. Display the delta-function basis for $V \oplus W$.

C. (optional) Show that the tensor-product vector space $V \otimes W$ is the same as (i.e., “canonically isomorphic” to) the free vector space on $S \times T$, i.e., the set of all ordered pairs (s,t) of elements $s \in S$ and $t \in T$. That is, construct an isomorphism between the two spaces, without resorting to choosing a basis.