Linear Systems, Black Boxes, and Beyond

Homework #1 (2014-2015), Answers

Q1: Impulse responses and transfer functions

A. Exponential decay: For a system F *with an impulse response* $f(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$*, find the transfer function* $\hat{f}(\omega)$.

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt = \int_{0}^{\infty} e^{-i\omega t} \lambda e^{-\lambda t} dt = \lambda \int_{0}^{\infty} e^{-(i\omega+\lambda)t} dt$$
$$= -\left(\frac{\lambda}{i\omega+\lambda}\right) e^{-(i\omega+\lambda)t}\Big|_0^{\infty} = \frac{\lambda}{i\omega+\lambda} = \frac{1}{1+i\omega/\lambda}$$

B. Pure delay: For a system F_T *with an impulse response* $f_T(t) = \delta(t-T)$*, find the transfer function* $\hat{f}_T(\omega)$.

$$\hat{f}_T(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f_T(t) dt = \int_{0}^{\infty} e^{-i\omega t} \delta(t-T) dt = e^{-i\omega T}$$

C. Differentiation: Consider a system F_{diff} *whose output is the derivative of the input. We can't write an impulse response for this system in a straightforward way, because the derivative of a delta-function is not defined. But we can determine its transfer function, by considering its response to sinusoids* $e^{i\omega t}$*. What is its transfer function* $\hat{f}_{diff}(\omega)$*?*

F_{diff} takes $e^{i\omega t}$ into $\frac{d}{dt} e^{i\omega t} = i\omega e^{i\omega t}$. So F_{diff} multiplies $e^{i\omega t}$ by $i\omega$. Therefore, $\hat{f}_{diff}(\omega) = i\omega$.

Note that $F_{diff} = \lim_{T\to 0} \frac{f_0 - f_T}{T}$ (the definition of the derivative). Correspondingly,

$$\lim_{T\to 0} \frac{\hat{f}_0(\omega) - \hat{f}_T(\omega)}{T} = \lim_{T\to 0} \frac{1-e^{-i\omega T}}{T} = \lim_{T\to 0} \frac{1-(1-i\omega T)}{T} = \lim_{T\to 0} \frac{i\omega T}{T} = i\omega = \hat{f}_{diff}(\omega)$$

Q2: Biased diffusion

In the notes, we modeled diffusion as a random walk from $x = 0$ *to* $x = \pm b$*, with equal probability, in time* ΔT*. That is,* $F_{\Delta T}(x) = \frac{1}{2}\left(\delta(x-b) + \delta(x+b)\right)$*. We saw that this had a stable limit as* $\Delta T \to 0$ *if* $b^2 = A\Delta T$*, i.e.,* $b = \sqrt{A\Delta T}$.

Now consider a biased process, in which the probability of a step to $+b$ *is* $\frac{1}{2}(1+\alpha)$ *and the probability of a step to* $-b$ *is* $\frac{1}{2}(1-\alpha)$*. So now,* $F_{\Delta T}(x) = \frac{1}{2}\left((1+\alpha)\delta(x-b) + (1-\alpha)\delta(x+b)\right)$.

A. Determine $\hat{F}_{\Delta T}(\omega)$.

$$\hat{F}_{\Delta T}(\omega)=\int_{-\infty}^{\infty}F_{\Delta T}(x)e^{-i\omega x}dx=\frac{1}{2}\int_{-\infty}^{\infty}\left((1+\alpha)\delta(x-b)+(1+\alpha)\delta(x+b)\right)e^{-i\omega x}dx$$
$$=\frac{1}{2}\int_{-\infty}^{\infty}\left(\delta(x-b)+\delta(x+b)\right)e^{-i\omega x}dx+\frac{\alpha}{2}\int_{-\infty}^{\infty}\left(\delta(x-b)-\delta(x+b)\right)e^{-i\omega x}dx$$
$$=\frac{1}{2}\left(e^{-i\omega b}+e^{i\omega b}\right)+\frac{\alpha}{2}\left(e^{-i\omega b}-e^{i\omega b}\right)=\cos(\omega b)-i\alpha\sin(\omega b)$$

B. How should α *vary with* ΔT *to ensure a stable limit for* $\hat{F}_T(\omega)=\lim_{\Delta T\to 0}\left(\hat{F}_{\Delta T}(\omega)\right)^{T/\Delta T}$ *as* $\Delta T\to 0$, *and what is this limit?*

For small b

$$\hat{F}_{\Delta T}(\omega)=\cos(\omega b)-i\alpha\sin(\omega b)\approx 1-\frac{1}{2}b^2\omega^2-i\alpha\omega b\approx e^{-\frac{1}{2}b^2\omega^2-i\alpha\omega b}.$$

So

$$\left(\hat{F}_{\Delta T}(\omega)\right)^{T/\Delta T}\approx e^{\left(-\frac{1}{2}b^2\omega^2-i\alpha\omega b\right)\frac{T}{\Delta T}}.$$

We need $b^2/\Delta T$ and $\alpha b/\Delta T$ to have a stable limit as $\Delta T\to 0$. So $b^2/\Delta T=A$ implies $b=\sqrt{A\Delta T}$, and $\alpha b/\Delta T=h$ implies $\alpha=h\Delta T/b=\dfrac{h\Delta T}{\sqrt{A\Delta T}}=h\sqrt{\dfrac{\Delta T}{A}}$.

So $\hat{F}_T(\omega)=\lim_{\Delta T\to 0}\left(\hat{F}_{\Delta T}(\omega)\right)^{T/\Delta T}=e^{\left(-\frac{1}{2}b^2\omega^2-i\alpha\omega b\right)\frac{T}{\Delta T}}=e^{-\omega^2AT/2-i\omega\alpha bT/\Delta T}=e^{-\omega^2AT/2-i\omega hT}$

C. If, at time 0, the distribution is $p_0(x)=\delta(x)$, *what is the distribution* $p_T(x)$ *at time* T *?*

$\hat{p}_T(\omega)=\hat{F}_T(\omega)\hat{p}(0)$, and since $p_0(x)=\delta(x)$, $\hat{p}(0)=1$. So $\hat{p}_T(\omega)=\hat{F}_T(\omega)=e^{-\omega^2AT/2+i\omega hT}$. Inverting the transform,

$$p_T(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{p}_T(\omega)e^{i\omega x}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\omega^2AT/2-i\omega hT}e^{i\omega x}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\omega^2AT/2}e^{i\omega(x-hT)}d\omega.$$

The final integral is the same as the one that arose for unbiased diffusion (notes), but with $x-hT$ replacing x.

So $p_T(x)=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\hat{p}_T(\omega)e^{i\omega x}d\omega==\dfrac{1}{\sqrt{2\pi AT}}e^{-(x-hT)^2/2AT}$, a Gaussian centered at $x=hT$ whose variance is $\sqrt{AT}$.

Not surprisingly, when the probabilities of leftward and rightward steps are unequal, the distribution drifts by an amount proportional to time.