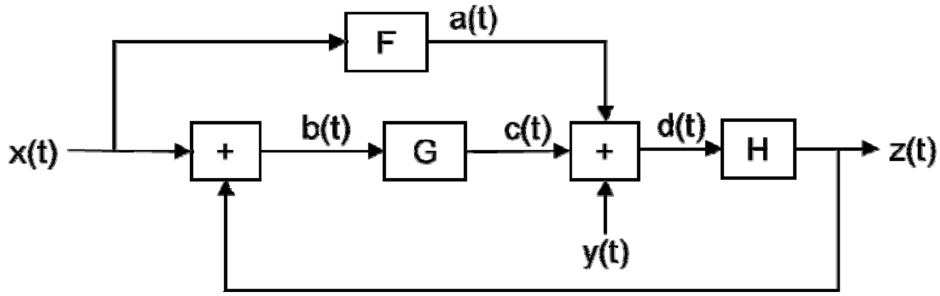
Linear Systems, Black Boxes, and Beyond

Homework #3 (2014-2015), Answers

Q1: Noises and networks

Given the following network, where F, G, and H are linear filters with transfer functions $\tilde{F}(\omega)$, $\tilde{G}(\omega)$, *and* $\tilde{H}(\omega)$, *and* $x(t)$ *and* $y(t)$ *are independent noise inputs with power spectra* $P_X(\omega)$ *and* $P_Y(\omega)$, *calculate the power spectrum* $P_Z(\omega)$ *of* $z(t)$.



We want to determine $P_Z(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle\left|\tilde{z}(\omega)\right|^2\right\rangle=\lim_{T\to\infty}\frac{1}{T}\left\langle\tilde{z}(\omega)\overline{\tilde{z}(\omega)}\right\rangle$, where $\tilde{z}(\omega)$ is a Fourier estimate of $z(t)$ over some finite but long interval *T*, i.e., $\tilde{z}(\omega)=\int_0^T e^{-i\omega t}z(t)dt$. To find $\tilde{z}(\omega)$ in terms of $\tilde{x}(\omega)$ and $\tilde{y}(\omega)$, we chase Fourier estimates through the network, considering *T* to be long enough so that the Fourier estimates can be replaced by the corresponding Fourier transforms:

$$\tilde{a}(\omega)=\tilde{F}(\omega)\tilde{x}(\omega)$$
$$\tilde{b}(\omega)=\tilde{x}(\omega)+\tilde{z}(\omega)$$
$$\tilde{c}(\omega)=\tilde{G}(\omega)\tilde{b}(\omega)=\tilde{G}(\omega)\tilde{x}(\omega)+\tilde{G}(\omega)\tilde{z}(\omega)$$
$$\tilde{d}(\omega)=\tilde{a}(\omega)+\tilde{c}(\omega)+\tilde{y}(\omega)=\tilde{F}(\omega)\tilde{x}(\omega)+\tilde{G}(\omega)\tilde{x}(\omega)+\tilde{G}(\omega)\tilde{z}(\omega)+\tilde{y}(\omega)$$
$$\tilde{z}(\omega)=\tilde{H}(\omega)\tilde{d}(\omega)=\tilde{H}(\omega)\tilde{F}(\omega)\tilde{x}(\omega)+\tilde{H}(\omega)\tilde{G}(\omega)\tilde{x}(\omega)+\tilde{H}(\omega)\tilde{G}(\omega)\tilde{z}(\omega)+\tilde{H}(\omega)\tilde{y}(\omega).$$

Solving the final equation for $\tilde{z}(\omega)$:

$$\tilde{z}(\omega)=\tilde{H}(\omega)\tilde{F}(\omega)\tilde{x}(\omega)+\tilde{H}(\omega)\tilde{G}(\omega)\tilde{x}(\omega)+\tilde{H}(\omega)\tilde{G}(\omega)\tilde{z}(\omega)+\tilde{H}(\omega)\tilde{y}(\omega),$$

so

$$\tilde{z}(\omega)\left(1-\tilde{H}(\omega)\tilde{G}(\omega)\right)=\tilde{H}(\omega)\tilde{F}(\omega)\tilde{x}(\omega)+\tilde{H}(\omega)\tilde{G}(\omega)\tilde{x}(\omega)+\tilde{H}(\omega)\tilde{y}(\omega),$$

so

$$\tilde{z}(\omega)=\frac{\tilde{H}(\omega)}{1-\tilde{H}(\omega)\tilde{G}(\omega)}\left(\left(\tilde{F}(\omega)+\tilde{G}(\omega)\right)\tilde{x}(\omega)+\tilde{y}(\omega)\right).$$

In computing $\left\langle\left|z(\omega)\right|^2\right\rangle=\left\langle\tilde{z}(\omega)\overline{\tilde{z}(\omega)}\right\rangle$, we will get terms involving just *x*, just *y*, and cross-terms between *x* and *y*. The expected value of the cross-terms vanish because *x* and *y* are hypothesized to be independent. So

$$\left\langle |z(\omega)|^2 \right\rangle = \left| \frac{\tilde{H}(\omega)}{1-\tilde{H}(\omega)\tilde{G}(\omega)} \right|^2 \left\langle \left| \left(\tilde{F}(\omega) + \tilde{G}(\omega) \right) \tilde{x}(\omega) + \tilde{y}(\omega) \right|^2 \right\rangle$$

$$= \left| \frac{\tilde{H}(\omega)}{1-\tilde{H}(\omega)\tilde{G}(\omega)} \right|^2 \left\langle \left(\left(\tilde{F}(\omega) + \tilde{G}(\omega) \right) \tilde{x}(\omega) + \tilde{y}(\omega) \right) \overline{\left(\left(\tilde{F}(\omega) + \tilde{G}(\omega) \right) \tilde{x}(\omega) + \tilde{y}(\omega) \right)} \right\rangle$$

$$= \left| \frac{\tilde{H}(\omega)}{1-\tilde{H}(\omega)\tilde{G}(\omega)} \right|^2 \left\langle \left(\left(\tilde{F}(\omega) + \tilde{G}(\omega) \right) \tilde{x}(\omega) \right) \overline{\left(\left(\tilde{F}(\omega) + \tilde{G}(\omega) \right) \tilde{x}(\omega) \right)} + \tilde{y}(\omega)\overline{\tilde{y}(\omega)} + cross - terms \right\rangle$$

$$= \left| \frac{\tilde{H}(\omega)}{1-\tilde{H}(\omega)\tilde{G}(\omega)} \right|^2 \left\langle \left| \tilde{F}(\omega) + \tilde{G}(\omega) \right|^2 \tilde{x}(\omega)\overline{\tilde{x}(\omega)} + \tilde{y}(\omega)\overline{\tilde{y}(\omega)} + cross - terms \right\rangle$$

$$= \left| \frac{\tilde{H}(\omega)}{1-\tilde{H}(\omega)\tilde{G}(\omega)} \right|^2 \left\langle \left| \tilde{F}(\omega) + \tilde{G}(\omega) \right|^2 \left| \tilde{x}(\omega) \right|^2 + \left| \tilde{y}(\omega) \right|^2 + cross - terms \right\rangle$$

$$= \left| \frac{\tilde{H}(\omega)}{1-\tilde{H}(\omega)\tilde{G}(\omega)} \right|^2 \left\langle \left| \tilde{F}(\omega) + \tilde{G}(\omega) \right|^2 \left| \tilde{x}(\omega) \right|^2 + \left| \tilde{y}(\omega) \right|^2 \right\rangle$$

So $\left\langle |z(\omega)|^2 \right\rangle = \left| \frac{\tilde{H}(\omega)}{1-\tilde{H}(\omega)\tilde{G}(\omega)} \right|^2 \left\langle \left| \tilde{F}(\omega) + \tilde{G}(\omega) \right|^2 \left| \tilde{x}(\omega) \right|^2 + \left| \tilde{y}(\omega) \right|^2 \right\rangle$.

Using $P_S(\omega) = \lim_{T\to\infty} \frac{1}{T} \left\langle \left| \tilde{s}(\omega) \right|^2 \right\rangle$,

$$P_Z(\omega) = \lim_{T\to\infty} \frac{1}{T} \left\langle |z(\omega)|^2 \right\rangle = \left| \frac{\tilde{H}(\omega)}{1-\tilde{H}(\omega)\tilde{G}(\omega)} \right|^2 \left[\left| \tilde{F}(\omega) + \tilde{G}(\omega) \right|^2 \lim_{T\to\infty} \frac{1}{T} \left\langle \left| \tilde{x}(\omega) \right|^2 \right\rangle + \lim_{T\to\infty} \frac{1}{T} \left\langle \left| \tilde{y}(\omega) \right|^2 \right\rangle \right]$$

$$= \left| \frac{\tilde{H}(\omega)}{1-\tilde{H}(\omega)\tilde{G}(\omega)} \right|^2 \left[\left| \tilde{F}(\omega) + \tilde{G}(\omega) \right|^2 P_X(\omega) + P_Y(\omega) \right].$$