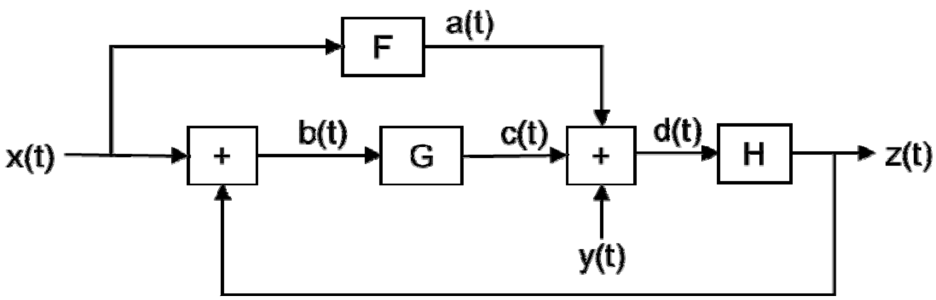
Linear Systems, Black Boxes, and Beyond

Homework #4 (2014-2015), Answers

Q1: Covariances in a network

[Same set-up as Q1 of Homework 3] Given the following network, where F, G, and H are linear filters with transfer functions $\tilde{F}(\omega)$, $\tilde{G}(\omega)$, and $\tilde{H}(\omega)$, and $x(t)$ and $y(t)$ are independent noise inputs with power spectra $P_X(\omega)$ and $P_Y(\omega)$:



A. Calculate the cross-spectra $P_{Z,X}(\omega)$ and $P_{Z,Y}(\omega)$.

(Referring to last week's homework), we have $\tilde{z}(\omega)=\tilde{K}(\omega)\left(\left(\tilde{F}(\omega)+\tilde{G}(\omega)\right)\tilde{x}(\omega)+\tilde{y}(\omega)\right)$, where

$\tilde{K}(\omega)=\dfrac{\tilde{H}(\omega)}{1-\tilde{H}(\omega)\tilde{G}(\omega)}$.

$P_{Z,X}(\omega)=\lim_{T\to\infty}\dfrac{1}{T}\left\langle \tilde{z}(\omega)\overline{\tilde{x}(\omega)}\right\rangle=\lim_{T\to\infty}\dfrac{1}{T}\left\langle \tilde{K}(\omega)\left(\left(\tilde{F}(\omega)+\tilde{G}(\omega)\right)\tilde{x}(\omega)+\tilde{y}(\omega)\right)\overline{\tilde{x}(\omega)}\right\rangle$. Since $x(t)$ and $y(t)$ are independent, $\left\langle \tilde{y}(\omega)\overline{\tilde{x}(\omega)}\right\rangle=0$, so

$$P_{Z,X}(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle \tilde{K}(\omega)\left(\tilde{F}(\omega)+\tilde{G}(\omega)\right)\tilde{x}(\omega)\overline{\tilde{x}(\omega)}\right\rangle$$
$$=\lim_{T\to\infty}\frac{1}{T}\tilde{K}(\omega)\left(\tilde{F}(\omega)+\tilde{G}(\omega)\right)\left\langle \tilde{x}(\omega)\overline{\tilde{x}(\omega)}\right\rangle \quad .$$
$$=\tilde{K}(\omega)\left(\tilde{F}(\omega)+\tilde{G}(\omega)\right)P_X(\omega)$$

Similarly,

$P_{Z,Y}(\omega)=\lim_{T\to\infty}\dfrac{1}{T}\left\langle \tilde{z}(\omega)\overline{\tilde{y}(\omega)}\right\rangle=\lim_{T\to\infty}\dfrac{1}{T}\left\langle \tilde{K}(\omega)\left(\left(\tilde{F}(\omega)+\tilde{G}(\omega)\right)\tilde{x}(\omega)+\tilde{y}(\omega)\right)\overline{\tilde{y}(\omega)}\right\rangle$. Again using $\left\langle \tilde{y}(\omega)\overline{\tilde{x}(\omega)}\right\rangle=0$,

$$P_{Z,Y}(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle \tilde{z}(\omega)\overline{\tilde{y}(\omega)}\right\rangle=\lim_{T\to\infty}\frac{1}{T}\left\langle \tilde{K}(\omega)\tilde{y}(\omega)\overline{\tilde{y}(\omega)}\right\rangle=\tilde{K}(\omega)\lim_{T\to\infty}\frac{1}{T}\left\langle \tilde{y}(\omega)\overline{\tilde{y}(\omega)}\right\rangle=\tilde{K}(\omega)P_Y(\omega).$$

B. Now assume that $x(t)$ and $y(t)$ are NOT independent, and their dependence is characterized by a nonzero cross-spectrum $P_{X,Y}(\omega)$. Calculate the power spectrum $P_Z(\omega)$ in terms of $P_X(\omega)$, $P_Y(\omega)$, and $P_{X,Y}(\omega)$.

We want to find $P_Z(\omega)=\lim_{T\to\infty}\dfrac{1}{T}\left\langle \left|\tilde{z}(\omega)\right|^2\right\rangle=\lim_{T\to\infty}\dfrac{1}{T}\left\langle \tilde{z}(\omega)\overline{\tilde{z}(\omega)}\right\rangle$ Again using

$\tilde{z}(\omega)=\tilde{K}(\omega)\left(\left(\tilde{F}(\omega)+\tilde{G}(\omega)\right)\tilde{x}(\omega)+\tilde{y}(\omega)\right)$,

$$\left\langle \tilde{z}(\omega)\overline{\tilde{z}(\omega)} \right\rangle = \left\langle \tilde{K}(\omega)\left(\left(\tilde{F}(\omega)+\tilde{G}(\omega)\right)\tilde{x}(\omega)+\tilde{y}(\omega)\right)\overline{\tilde{K}(\omega)\left(\left(\tilde{F}(\omega)+\tilde{G}(\omega)\right)\tilde{x}(\omega)+\tilde{y}(\omega)\right)} \right\rangle$$

$$= \tilde{K}(\omega)\overline{\tilde{K}(\omega)} \bullet$$

$$\left\langle \left(\tilde{F}(\omega)+\tilde{G}(\omega)\right)\tilde{x}(\omega)\overline{\left(\tilde{F}(\omega)+\tilde{G}(\omega)\right)\tilde{x}(\omega)} + \left(\tilde{F}(\omega)+\tilde{G}(\omega)\right)\tilde{x}(\omega)\overline{\tilde{y}(\omega)} + \tilde{y}(\omega)\overline{\left(\tilde{F}(\omega)+\tilde{G}(\omega)\right)\tilde{x}(\omega)} + \tilde{y}(\omega)\overline{\tilde{y}(\omega)} \right\rangle.$$

$$= \left|\tilde{K}(\omega)\right|^2 \bullet$$

$$\left\langle \left|\tilde{F}(\omega)+\tilde{G}(\omega)\right|^2 \tilde{x}(\omega)\overline{\tilde{x}(\omega)} + \left(\tilde{F}(\omega)+\tilde{G}(\omega)\right)\tilde{x}(\omega)\overline{\tilde{y}(\omega)} + \overline{\left(\tilde{F}(\omega)+\tilde{G}(\omega)\right)}\tilde{y}(\omega)\overline{\tilde{x}(\omega)} + \tilde{y}(\omega)\overline{\tilde{y}(\omega)} \right\rangle$$

So

$$P_Z(\omega) = \lim_{T\to\infty} \frac{1}{T}\left\langle \tilde{z}(\omega)\overline{\tilde{z}(\omega)} \right\rangle$$

$$= \left|\tilde{K}(\omega)\right|^2 \bullet$$

$$\lim_{T\to\infty} \frac{1}{T}\left\langle \left|\tilde{F}(\omega)+\tilde{G}(\omega)\right|^2 \tilde{x}(\omega)\overline{\tilde{x}(\omega)} + \left(\tilde{F}(\omega)+\tilde{G}(\omega)\right)\tilde{x}(\omega)\overline{\tilde{y}(\omega)} + \overline{\left(\tilde{F}(\omega)+\tilde{G}(\omega)\right)}\tilde{y}(\omega)\overline{\tilde{x}(\omega)} + \tilde{y}(\omega)\overline{\tilde{y}(\omega)} \right\rangle.$$

$$= \left|\tilde{K}(\omega)\right|^2 \left(\left|\tilde{F}(\omega)+\tilde{G}(\omega)\right|^2 P_X(\omega) + \left(\tilde{F}(\omega)+\tilde{G}(\omega)\right)P_{X,Y} + \overline{\left(\tilde{F}(\omega)+\tilde{G}(\omega)\right)}\overline{P_{X,Y}} + P_Y(\omega) \right)$$

Q2. Multiple signals with common and private noise sources

Say there are N observed signals $z_i(t)$, each of which is the result of adding a common noise source $x(t)$, filtered by a linear filter F_i, to a private noise source $y_i(t)$, filtered by a linear filter G_i. All the noises $x(t)$ and $y_i(t)$ are assumed independent.

A. Determine the cross-spectra P_{Z_i,Z_j} in terms of the power spectra P_X, P_{Y_i}, and the filter characteristics $\tilde{F}_i$ and $\tilde{G}_i$.

The Fourier estimates $\tilde{z}_i(\omega)$ are given by $\tilde{z}_i(\omega) = \tilde{F}_i(\omega)\tilde{x}(\omega) + \tilde{G}_i(\omega)\tilde{y}(\omega)$, so $\left\langle \tilde{z}_i(\omega)\overline{\tilde{z}_j(\omega)} \right\rangle = \left\langle \left(\tilde{F}_i(\omega)\tilde{x}(\omega)+\tilde{G}_i(\omega)\tilde{y}_i(\omega)\right)\left(\overline{\tilde{F}_j(\omega)\tilde{x}(\omega)+\tilde{G}_j(\omega)\tilde{y}_j(\omega)}\right) \right\rangle$. Using independence of all of the noises, $\left\langle \tilde{z}_i(\omega)\overline{\tilde{z}_j(\omega)} \right\rangle = \left\langle \tilde{F}_i(\omega)\tilde{x}(\omega)\overline{\tilde{F}_j(\omega)\tilde{x}(\omega)} \right\rangle = \tilde{F}_i(\omega)\overline{\tilde{F}_j(\omega)}\left\langle \tilde{x}(\omega)\overline{\tilde{x}(\omega)} \right\rangle$.

So (for $i \neq j$),

$$P_{Z_i,Z_j}(\omega) = \lim_{T\to\infty} \frac{1}{T}\left\langle \tilde{z}_i(\omega)\overline{\tilde{z}_j(\omega)} \right\rangle = \lim_{T\to\infty} \frac{1}{T}\tilde{F}_i(\omega)\overline{\tilde{F}_j(\omega)}\left\langle \tilde{x}(\omega)\overline{\tilde{x}(\omega)} \right\rangle = \tilde{F}_i(\omega)\overline{\tilde{F}_j(\omega)}P_X(\omega).$$

For $i = j$, the cross-terms involving y do not vanish, so

$$P_{Z_i,Z_i}(\omega) = \lim_{T\to\infty} \frac{1}{T}\left\langle \tilde{z}_i(\omega)\overline{\tilde{z}_i(\omega)} \right\rangle = \lim_{T\to\infty} \frac{1}{T}\left(\tilde{F}_i(\omega)\overline{\tilde{F}_i(\omega)}\left\langle \tilde{x}(\omega)\overline{\tilde{x}(\omega)} \right\rangle + \tilde{G}_i(\omega)\overline{\tilde{G}_i(\omega)}\left\langle \tilde{y}_i(\omega)\overline{\tilde{y}_i(\omega)} \right\rangle \right)$$

$$= \left|\tilde{F}_i(\omega)\right|^2 P_X(\omega) + \left|\tilde{G}_i(\omega)\right|^2 P_{Y_i}(\omega).$$

B. Now assume that all of the private noises $y_i(t)$ are 0. Consider, for each frequency ω, the matrix $P_{Z_i,Z_j}(\omega)$. Does it have any special properties?

It has only one nonzero eigenvalue. To see this, note that $P_{Z_i,Z_j}(\omega) = \tilde{F}_i(\omega)\overline{\tilde{F}_j(\omega)}P_X(\omega)$, even if $i = j$. So the rows of $P_{Z_i,Z_j}(\omega)$ are all common multiples of each other. (The same holds for columns). That is, $P_{Z_i,Z_j}(\omega)$ is the product of a column matrix, consisting of the $\tilde{F}_i(\omega)$, and a row matrix, consisting of the $\overline{\tilde{F}_j(\omega)}$, and the constant factor P_X. So, as a linear transformation, its range is a one-dimensional space, so it can have at most one nonzero eigenvector.

The corresponding eigenvalue is the trace of $P_{Z_i,Z_j}(\omega)$ (since the trace is always the sum of the eigenvalues).

$$tr\left(P_{Z_i,Z_j}(\omega)\right) = \sum_i P_{Z_i,Z_i}(\omega) = P_X(\omega)\sum_i \left|\tilde{F}_i(\omega)\right|^2 .$$

Note: the "global coherence" is defined as the ratio of the largest eigenvalue of the cross-spectral matrix $P_{Z_i,Z_j}(\omega)$ to its trace. In the scenario in which all signals have a shared noise (and no private noise sources), we have just shown that the global coherence is 1.