Linear Transformations and Group Representations

Homework #2 (2014-2015), Answers

Q1: Properties of self-adjoint and unitary operators

A. Say A and B are both self-adjoint. Is $A+B$ self-adjoint?
Yes. $\langle (A+B)v, w\rangle = \langle Av, w\rangle + \langle Bv, w\rangle = \langle v, A^*w\rangle + \langle v, B^*w\rangle$.
Since A and B are both self-adjoint,
$\langle v, A^*w\rangle + \langle v, B^*w\rangle = \langle v, Aw\rangle + \langle v, Bw\rangle = \langle v, (A+B)w\rangle$.
Putting these two lines together, $\langle (A+B)v, w\rangle = \langle v, (A+B)w\rangle$.
B. Say A and B are both self-adjoint. Is AB self-adjoint?
Not necessarily.
$\langle ABv, w\rangle = \langle Bv, A^*w\rangle = \langle v, B^*A^*w\rangle = \langle v, BAw\rangle$. So AB is self-adjoint only if $AB = BA$.

C. Say A and B are both unitary. Is $A+B$ unitary?
Not necessarily. We would need $(A+B)^* = (A+B)^{-1}$, or $(A+B)^*(A+B) = I$. But $(A+B)^*(A+B) = A^*A + A^*B + B^*A + B^*B = I + A^{-1}B + B^{-1}A + I$, so $(A+B)^*(A+B) = I$ requires $A^{-1}B + B^{-1}A + I = 0$, which is not true in general (consider for example $A = B = I$).

D. Say A and B are both unitary. Is AB unitary?
Yes. $\langle ABv, w\rangle = \langle Bv, A^*w\rangle = \langle Bv, A^{-1}w\rangle = \langle v, B^*A^{-1}w\rangle = \langle v, B^{-1}A^{-1}w\rangle = \langle v, (AB)^{-1}w\rangle$, so the adjoint of AB is $(AB)^{-1}$, as required for AB to be unitary. Note that this means that the unitary invertible of $Hom(V,V)$ form a group.

Q2. Time translation is unitary

Recall that the time translation operator D_T is defined by $(D_T v)(t) = v(t+T)$. Show that D_T is unitary.

We showed that $(D_T)^* = D_{-T}$, and also, that $D_S D_T = D_{S+T}$. So $D_T D_{-T} = I$, i.e., $D_T (D_T)^* = I$,

So D_T and $(D_T)^* = D_{-T}$ are inverses, as required for D_T to be unitary.

Q3. Relationship between unitary and self-adjoint operators.

A. Say A is self-adjoint. Show that $(iA)^* = -(iA)$.

$\langle iAv, w\rangle = i\langle Av, w\rangle = i\langle v, A^*w\rangle = i\langle v, Aw\rangle = \langle v, \bar{i}Aw\rangle = \langle v, -iAw\rangle$. So $(iA)^* = -(iA)$.

B. Say A is self-adjoint. Show that $U = e^{iA}$ is unitary. Do this by considering the formal power series definition $e^M = \sum_{j=0}^{\infty} \frac{1}{j!} M^j$.

Using the formal power series definition $e^M = \sum_{j=0}^{\infty} \frac{1}{j!} M^j$ with $M = iA$, we have

$U = e^{iA} = \sum_{j=0}^{\infty} \frac{1}{j!} (iA)^j$. So $\left(e^{iA}\right)^* = \sum_{j=0}^{\infty} \frac{1}{j!} \left((iA)^j\right)^* = \sum_{j=0}^{\infty} \frac{1}{j!} \left((iA)^*\right)^j$. From part A, $(iA)^* = -(iA)$.

So

$$U^* = \left(e^{iA}\right)^* = \sum_{j=0}^{\infty} \frac{1}{j!} \left((iA)^*\right)^j = \sum_{j=0}^{\infty} \frac{1}{j!} \left(-iA\right)^j = e^{-iA} = \left(e^{iA}\right)^{-1} = U^{-1}.$$