Linear Transformations and Group Representations

Homework #3 (2014-2015), Answers

Representations of the dihedral group of D_4

Here we construct some representations of the dihedral group D_4, i.e., the rotations and reflections of a square. (These are not all of its representations, and these are not necessarily irreducible.) Let's use the following notation for its elements:

I : the identity
R, R^{-1} : 90-degree rotations right and left ($R^4 = I$)
C : rotation by 180 deg ($C^2 = I$, $R^2 = C$)
X , Y : mirror flips in the x- and y-axes ($X^2 = Y^2 = 1$)
$M_\backslash$, $M_/$: mirror flips on the two diagonals ($M_\backslash{}^2 = M_/{}^2 = I$).

Here is the start of a character table, i.e., $\chi_L(g) = tr(L_g)$, where g is one of the $g \in \{I, R, R^{-1}, C, X, Y, M_\backslash, M_/\}$. Note that for group elements g and h that are intrinsically identical, i.e., related by $h = \alpha g \alpha^{-1}$, then $\chi_L(g) = \chi_L(h)$, so we've grouped them into a single column. For example, $R^{-1} = XRX^{-1}$, $Y = M_\backslash X M_\backslash{}^{-1}$, and $M_\backslash = XM_/X^{-1}$ (try it!).

When L is the trivial representation, L_g is the 1×1 identify matrix, whose trace is 1:

			g		
L	I	R , R^{-1}	C	X , Y	$M_\backslash$, $M_/$
trivial representation	1	1	1	1	1

A. Determine the characters for the representation based on 2×2 matrices that express the rotations and mirror-flips of the square in the standard coordinate plane.

$$I \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, R \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, C \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, X \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_\backslash \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

L	I	R , R^{-1}	C	X , Y	$M_\backslash$, $M_/$
Trivial rep	1	1	1	1	1
2×2 rep	2	0	-2	0	0

B. Determine the characters based on permutation matrices, where we consider the four corners to be the objects permuted.

$$I \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, R \to \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, C \to \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, X \to \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad M_\backslash \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

L	I	R, R^{-1}	C	X, Y	$M_\backslash$, $M_/$
Trivial rep	1	1	1	1	1
2×2 rep	2	0	-2	0	0
Corner perm	4	0	0	0	2

C. Determine the characters based on the permutation matrices, where we consider the four sides to be the objects permuted.

$$I \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, R \to \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, C \to \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, X \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad M_\backslash \to \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

L	I	R, R^{-1}	C	X, Y	$M_\backslash$, $M_/$
Trivial rep	1	1	1	1	1
2×2 rep	2	0	-2	0	0
Corner perm	4	0	0	0	2
Side perm	4	0	0	2	0

D. Determine the characters based on the permutation matrices, where we consider two diagonals to be the objects permuted.

$$I \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, R \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, C \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, X \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_\backslash \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

L	I	R, R^{-1}	C	X, Y	$M_\backslash$, $M_/$
L	I	R, R^{-1}	C	X, Y	$M_\backslash$, $M_/$
Trivial rep	1	1	1	1	1
2×2 rep	2	0	-2	0	0
Corner perm	4	0	0	0	2
Side perm	4	0	0	2	0
Diag perm	2	0	2	0	2

E. Which of the above representations contain the trivial (identity) representation?

All but the 2×2 representation, since for each of those, $\frac{1}{|G|}\sum_g \chi_L(g) = 1$.