Linear Transformations and Group Representations

Homework #4 (2014-2015), Questions

Q1. Irreducible representations of the dihedral group of D_4

Here we build on the homework from last week to construct all the irreducible representations of dihedral group D_4, i.e., the rotations and reflections of a square. We continue to use the following notation for its elements:

I : the identity
R, R^{-1}: 90-degree rotations right and left ($R^4 = I$)
C : rotation by 180 deg ($C^2 = I$, $R^2 = C$)
X , Y : mirror flips in the x- and y-axes ($X^2 = Y^2 = 1$)
$M_\backslash$, $M_/$: mirror flips on the two diagonals ($M_\backslash{}^2 = M_/{}^2 = I$).

We had the following table of characters – the last line added in class and is the representation that maps a group element to +1 or -1 depending on whether it exchanges the front and back faces of the square:

L	I	R, R^{-1}	C	X , Y	$M_\backslash$, $M_/$
L_I : Trivial rep	1	1	1	1	1
$L_{2\times 2}$: 2×2 matrices	2	0	-2	0	0
L_{corner} : Corner perm	4	0	0	0	2
L_{side} : Side perm	4	0	0	2	0
L_{diag} : Diag perm	2	0	2	0	2
L_{face} : Face exchange	1	1	1	-1	-1

We also determined that L_{corner} , L_{side} , and L_{diag} contained one copy of the trivial representation, since for each of those, $\frac{1}{|G|}\sum_g \chi_L(g) = 1$.

A. For the representations L that contain the trivial representation, replace their entries in the above character table with the characters of the smaller representations L' for which $L = L' \oplus L_I$.

B. Using the Group Representation Theorem characterization that for irreducible representations, characters are orthonormal, identify the representations that are reducible.

C. Recall that $d(L,M) = \frac{1}{|G|}\sum_g \overline{\chi_L(g)}\chi_M(g)$ indicates how ways that an irreducible piece of a representation L can be matched to an irreducible piece of a representation M . So if L is irreducible, it indicates how many copies of L are inside of M . Use this to further reduce the remaining reducible representations.

D. Show that the table now has all of the irreducible representations of the dihedral group.

Q2. Representations of subgroups: an irreducible representation may become reducible, when restricted to a subgroup.

Setup: A representation L of a group G is, necessarily, a representation for any subgroup H of G, simply by restricting it to $g \in H$. But if a representation is irreducible on G, it need not be irreducible on H. A trivial example of this is to start with a representation of dimension $d > 1$, and restrict it to the one-element identity subgroup of G; in this case, the representation maps the identity element to the $d \times d$ identity matrix – which clearly is reducible. But here's a less-trivial example that illustrates what is more generic.

We consider the cyclic group $\mathbb{Z}_4$, which is the rotation group of the square – and hence, a subgroup of D_4 considered in Q1.As in the class notes, $\mathbb{Z}_n$ it has a representation L_m for every nth root of unity, which takes a $2\pi/n$ rotation to $\exp(\frac{2\pi i}{n}m)$. Here $n = 4$, and we adopt the notation of Q1, so R is a rotation by $\pi/2$, $R^{-1} = R^3$ is a rotation by $3\pi/2$,and $C = R^2$ is a rotation by π. So the character table of $\mathbb{Z}_n$ is

L	I	R	$R^2 = C$	$R^3 = R^{-1}$
L_0 $(m = 0)$	1	1	1	1
L_1 $(m = 1)$	1	i	-1	-i
L_2 $(m = 2)$	1	-1	1	-1
L_3 $(m = 3)$	1	-i	-1	i

Now consider the irreducible representations of D_4, determined in Q1. Find their characters, considered as a representation of $\mathbb{Z}_4$. Which ones are reducible, and which are irreducible? How do they relate to the above irreducible representations of $\mathbb{Z}_4$?