Exam, 2014-2015

Do a total of 30 points (more if you want, of course). Show your work!

Q1: 10 points (4 parts, 2 points each for the first two parts, 3 points each for the second two parts)
Q2: 10 points (11 parts, 1 point each, max credit 10 points)
Q3: 10 points (5 parts, 2 points each)
Q4 :12 points (6 parts, 2 points each)

Q1: Constructing a group representation based on cosets.

Let G be a finite group, and *H* a subgroup of *G*. Recall that the coset *Hb* is the set of all of the elements *g* of *G* that can be written in the form $g = h \circ b$, for some element *h* in *H*, and that the cosets constitute a partition of *G* into disjoint subsets.

Now consider the free vector space *W* on the cosets: that is, *W* be the vector space of functions from the cosets *Hb* to $\mathbb{C}$. Since W is a free vector space, it has a natural inner product, $\langle x, y \rangle = \sum_{\text{cosets } Hb} x(Hb)\overline{y(Hb)}$,

For any function *x* in *W* and any element *p* of *G*, we define Q_p, a member of $Hom(W,W)$, as follows: Q_p takes *x* (a function on the cosets) to the $Q_p(x)$ (another function on the cosets) whose value at the coset *Hb* is given by $\left(Q_p(x)\right)(Hb) = x(Hbp)$.

A. Show that this is a group representation.
B. What is its dimension (in terms of the sizes $\#(G)$ of *G*, and $\#(H)$ of *H*)?
C. Suppose further that *G* is commutative. What is the character of *Q*?
D. (No longer supposing that *G* is commutative). Under what circumstances is *Q* irreducible?

Solutions
This set-up generalizes the "regular representation" (it reduces to the regular representation when *H* is the identity subgroup), and the analysis proceeds along the same lines.

A. To see that Q_p is unitary:

$$\left\langle Q_p(x), Q_p(y) \right\rangle = \sum_{\text{cosets } Hb} \left(Q_p(x)\right)(Hb)\overline{\left(Q_p(y)\right)(Hb)} = \sum_{\text{cosets } Hb} x(Hbp)\overline{y(Hbp)} = \sum_{\text{cosets } Hc} x(Hc)\overline{y(Hc)} = \langle x, y \rangle.$$

The crucial observation is the third equality: for a fixed group element *p*, if *Hb* ranges over all cosets, sampling each one exactly once, then so does *Hbp*. To see this, note that if $Hb'p$ and $Hb''p$ overlap, then some $h'b'p = h''b''p$, from which it follows that $h'b' = h''b''$, and that Hb' and Hb'' overlap also; since the cosets are disjoint, if two overlap, they must be identical. Formally, this is a change of variables from *b* to $c = bp$; $b = cp^{-1}$ if cp^{-1} samples each coset once, then so does *b*.

To see that Q_p is a representation – i.e., that $Q_pQ_q = Q_{pq}$, we need to show that $Q_p\left(Q_q(x)\right) = Q_{pq}(x)$ by evaluating the left and right hand side at every coset *Hb*. We do this just as in the analysis of the regular representation. On the left, say $y = Q_q(x)$, so $y(Hb) = \left(Q_q(x)\right)(Hb) = x(Hbq)$. Then $\left(Q_p(y)\right)(Hb) = y(Hbp) = x(Hbpq)$. On the right, $\left(Q_{pq}(x)\right)(Hb) = x(Hbpq)$ directly from the definition of *Q*.

B. Since the cosets divide a group into disjoint subsets, there are $\#(G)/\#(H)$ cosets. This is the dimension of the free vector space, since the free vector space consists of all the functions on these cosets, and therefore is the dimension of the representation.

For C and D: The character is defined by $\chi_Q(g)=\mathrm{tr}(Q_g)$. It is easiest to evaluate the trace using the basis for the free vector space *W* consisting of functions φ_{Hd} which assign a value of 1 to one coset *Hd*, and a value of 0 to other cosets. (This is also parallel to the situation for the regular representation, where we evaluate the trace in the free vector space on *G*, rather than in the free vector space on the cosets). In this basis, *Q* acts as a permutation matrix. To see this: $Q_g\left(\varphi_{Hd}(Hb)\right)=\varphi_{Hd}(Hbg)$. This evaluates to 1 if *Hbg* and *Hd* are the same coset, i.e., if *Hb* and Hdg^{-1} are the same coset. So $Q_g\left(\varphi_{Hd}\right)=\varphi_{Hdg^{-1}}$, i.e., an element of the basis set. Since the trace is the sum of the diagonal element entries in the matrix that describes the transformation, the character $\chi_Q(g)$ is the tally of the number of cosets *Hd* for which Hd and Hdg^{-1} are identical.

C. For a commutative group, the cosets Hdg^{-1} and $Hg^{-1}d$ are identical (since g^{-1} and *d* commute). So if $g\in H$, Q_g maps every φ_{Hd} to itself , since $Hdg^{-1}=Hg^{-1}d=Hd$ (the last step because $g\in H$). That is, for $g\in H$, Q_g, is the identity transformation. ,So for $g\in H$, $\chi_Q(g)=\mathrm{tr}(Q_g)=\dim W=\#(G)/\#(H)$. Conversely, if $g\notin H$, then $Hg^{-1}d$ and Hd must be distinct cosets, (since if they overlapped, then for some *h* and h', we would have $hg^{-1}d=h'd$ which would imply $hg^{-1}=h'$ and that $g\in H$). So if $g\notin H$, Q_g maps every φ_{Hd} to a different basis element, and therefore $\chi_Q(g)=\mathrm{tr}(Q_g)=0$. So in the case that *G* is commutative, $\chi_Q(g)=\#(G)/\#(H)$ for $g\in H$ and 0 otherwise.

D. As above, the character is always a non-negative integer – since it is the trace of a permutation matrix. The trace formula for the number of common irreducible components of *L* and *M* times is $d(L,M)=\frac{1}{|G|}\sum_g\overline{\chi_L(g)}\chi_M(g)$. Take $L=Q$ and $M=E$, the trivial representation. $\chi_E(g)=1$ everywhere, and $\chi_Q(g)\geq 0$ throughout *G*, with $\chi_Q(e)=\#(G)/\#(H)>0$. So $d(Q,E)>0$. So *Q* must contain at least one copy of the trivial representation. So the only way that *Q* can be irreducible is if it is, itself, the trivial representation, which means that it has dimension 1, so $\#(H)=\#(G)$, so $H=G$.

Question 2. Permutation matrices and eigendecompositions

Consider an $n\times n$ cyclic permutation matrix *M*, defined by $m_{j,j+1}=1$ for $1\leq j\leq(n-1)$, $m_{n,1}=1$, and otherwise zero. Here, for $n=5$: $M=\begin{pmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\1&0&0&0&0\end{pmatrix}$.

A. Is *M* unitary?
B. Is *M* self-adjoint?
C. What are the eigenvalues of *M*?
D. What are the eigenvectors of *M*?

E. Consider a matrix C, which is cyclic but with entries not necessarily drawn from $\{0,1\}$. (so it is not a permutation matrix). Specifically, $c_{j,k} = f_{k-j}$, where $k-j$ is interpreted mod n. For example, for $n=5$,

$$C = \begin{pmatrix} f_0 & f_1 & f_2 & f_3 & f_4 \\ f_4 & f_0 & f_1 & f_2 & f_3 \\ f_3 & f_4 & f_0 & f_1 & f_2 \\ f_2 & f_3 & f_4 & f_0 & f_1 \\ f_1 & f_2 & f_3 & f_4 & f_0 \end{pmatrix}.$$ Is C unitary?

F. Is C self-adjoint?
G. Does C commute with M?
H. What are the eigenvectors of C?
I. What are the eigenvalues of C?
J. What are the eigenvalues of this matrix, which is a permutation matrix but not cyclic?

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

K. What can you say about the eigenvalues of any permutation matrix of size $n \times n$?

Solutions

A. By direct calculation, $MM^T = 1$ ($M^* = M^T$ since M is real), so M is unitary.
B. Except for $n \le 2$, $m_{2,1} \ne m_{1,2}$ so M is not self-adjoint.

C and D. Say $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}$ is an eigenvector. Then $M\vec{x} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ x_1 \end{pmatrix}$, and $M\vec{x} = \lambda\vec{x}$ means that $x_2 = \lambda x_1$,

$x_3 = \lambda x_2 = \lambda^2 x_1$, …, $x_n = \lambda x_{n-1} = \lambda^{n-1} x_1$, and finally $x_1 = \lambda x_n = \lambda^n x_1$. So $\lambda^n = 1$, and x_1 is free to vary. That is, λ is an nth root of unity. For any such λ, i.e., for $\lambda_r = e^{\frac{2\pi i}{n} r}$, the eigenvectors are given by $\vec{x}_r = a \begin{pmatrix} 1 \\ \lambda_r \\ \lambda_r^{\,2} \\ \vdots \\ \lambda_r^{\,n-2} \\ \lambda_r^{\,n-1} \end{pmatrix}$.

E. Typically not, as being unitary $(CC^* = I)$ would require that $\sum_{j=1}^{n} f_j^{\,2} = 1$ (among other things).

F. Typically not, as $C = C^*$ would require $f_1 = \overline{f_{n-1}}$ (among other things).

G. Yes (by direct calculation): $MC = CM$. Note that MC is a matrix similar in structure to M but now f_{n-1} is on the diagonal.
H. Since M and C commute, the eigenvectors for C are the eigenvectors for M.

I. Choose $a = 1$ in the above expression (part D) for $\vec{x}_r$ and $\lambda_r = e^{\frac{2\pi i}{n}r}$. $M\vec{x}_r = M\begin{pmatrix} 1 \\ \lambda_r \\ \lambda_r^{\,2} \\ \vdots \\ \lambda_r^{\,n-2} \\ \lambda_r^{\,n-1} \end{pmatrix}$; the first entry of the product just be the first eigenvalue; by direct multiplication, this is $\sum_{j=0}^{n-1} f_j e^{\frac{2\pi i}{n}rj}$.

J. P is a block matrix $P = \begin{pmatrix} M_3 & 0 \\ 0 & M_4 \end{pmatrix}$. That is, it acts separately on a 3-dimensional and a 4-dimensional subspace, and in each of these subspaces, it is a cyclic permutation matrix of the sort defined in part A. On the 3-dimensional subspace, its eigenvalues are the same as those of M above with $n = 3$; on the 4-dimesional subspace, its eigenvalues are the same as of M above with $n = 4$.

K. They will always be rth roots of unity for some $r <= n$.

Question 3. Transfer function and power spectrum in a simple physical system

Say an output, $x(t)$, is related to an input, $s(t)$ by $m\frac{d^2x}{dt^2} = s - kx - c\frac{dx}{dt}$. That is, the output x is the position of an object of mass m that is subject to a sum of three forces: the input signal s, a spring-like restoring force $-kx$, and a frictional force $-c\frac{dx}{dt}$.

A. Determine the transfer function that relates x to s.

B. Say s is white noise. What is the power spectrum of x?

C and D. Under what circumstances does the power spectrum have a peak at a nonzero frequency? At what frequency is the peak?
E. Describe what happens if $c = 0$ (no friction).

Solutions

A. The relationship between $x(t)$ and $s(t)$ is linear and time-translation-invariant, so it preserves eigenfunctions of the time-translation operator, namely, the exponentials $e^{i\omega t}$. So if $s(t) = e^{i\omega t}$, then $x(t)$ must be a multiple of $s(t)$, i.e., $x(t) = \hat{L}(\omega)e^{i\omega t}$. Then $\frac{dx}{dt} = i\omega\hat{L}(\omega)e^{i\omega t}$ and $\frac{d^2x}{dt^2} = (i\omega)^2\,\hat{L}(\omega)e^{i\omega t} = -\omega^2\hat{L}(\omega)e^{i\omega t}$.
Substituting for each of the terms in $m\frac{d^2x}{dt^2} = s - kx - c\frac{dx}{dt}$: $m\left(-\omega^2\hat{L}(\omega)e^{i\omega t}\right) = e^{i\omega t} - k\hat{L}(\omega)e^{i\omega t} - ic\omega\hat{L}(\omega)e^{i\omega t}$.
Solving for $\hat{L}(\omega)$ yields $\hat{L}(\omega)\left(-m\omega^2 + k + ic\omega\right)\left(e^{i\omega t}\right) = e^{i\omega t}$ or, $\hat{L}(\omega) = \frac{1}{-m\omega^2 + k + ic\omega}$.

B. In general, the input and output power spectra of a linear system are related by $P_X(\omega) = \left|\tilde{L}(\omega)\right|^2 P_S(\omega)$.

Taking $P_S(\omega) = K$, $P_X(\omega) = K\left|\tilde{L}(\omega)\right|^2 = K\left|\dfrac{1}{-m\omega^2 + k + ic\omega}\right|^2$.

$$P_X(\omega) = K\left(\frac{1}{-m\omega^2 + k + ic\omega}\right)\overline{\left(\frac{1}{-m\omega^2 + k + ic\omega}\right)}$$

This is $= K\left(\dfrac{1}{-m\omega^2 + k + ic\omega}\right)\left(\dfrac{1}{-m\omega^2 + k - ic\omega}\right)$.

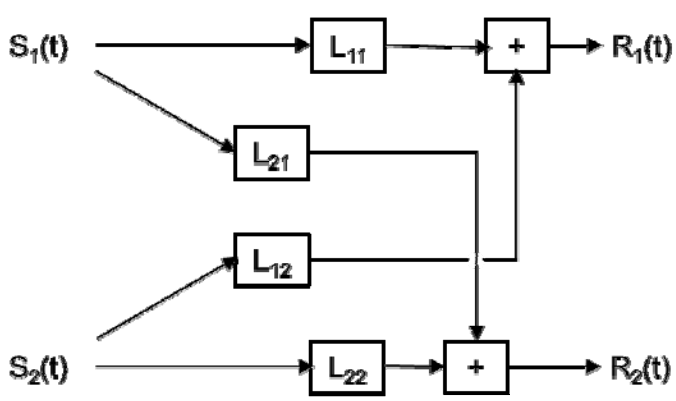
$$= K\left(\frac{1}{\left(-m\omega^2 + k\right)^2 + c^2\omega^2}\right) = K\left(\frac{1}{m^2\omega^4 + \left(c^2 - 2mk\right)\omega^2 + k^2}\right)$$

C and D. The power spectrum has a peak at a nonzero frequency when the denominator above has a minimum at a nonzero frequency. The denominator is a quadratic in ω^2, and in general, the quadratic $Au^2 + Bu + C$ (with $A > 0$) has an extremum at $u = -\dfrac{B}{2A}$. Here, $A = m^2$ and $B = c^2 - 2mk$, so there is a minimum of the denominator (and a peak of the power spectrum) at $\omega^2 = \dfrac{2mk - c^2}{2m^2}$. This is positive for $c < \sqrt{2mk}$, so if $c < \sqrt{2mk}$, there is a peak in the power spectrum at a positive frequency. (If this quantity is negative, then there is no real $\omega > 0$ at which there is a peak). The peak is at the frequency $\omega = \sqrt{\dfrac{2mk - c^2}{2m^2}}$.

E. When $c = 0$, there is a frequency $\omega = \sqrt{\dfrac{k}{m}}$ at which $\tilde{L}(\omega)$ and $P_X(\omega)$ are both infinite – i.e., at which an infinitesimal input will lead to an arbitrarily large response. That is, there is a resonance.

Question 4. Coherence and network identification

Say $S_1(t)$ and $S_2(t)$ are noise sources with power spectra $P_{S_1}(\omega)$ and $P_{S_2}(\omega)$, not necessarily independent, which are connected to two observable outputs $R_1(t)$ and $R_2(t)$ by the following network,where L_{ij} are linear filters with transfer functions $\tilde{L}_{ij}(\omega)$.



A. Find the power spectra $P_{R_1}(\omega)$ and $P_{R_2}(\omega)$ of the outputs in terms of the power spectra $P_{S_i}(\omega)$ and cross-spectrum $P_{S_1,S_2}(\omega)$ of the inputs.

B. Find the cross-spectrum of the outputs.

C and D. Same as A and B, but for *N* independent noise sources fully connected to *N* observable outputs, rather than two.

E. Define $S(\omega)$ as the "cross-spectral matrix" of S, i.e., the matrix whose elements are given by S

$S_{ij}(\omega)=\begin{cases} P_{S_i}(\omega), & i=j \\ P_{S_i,S_j}(\omega), & i\neq j \end{cases}$, and similarly for R. Further define $\tilde{L}(\omega)$ as the matrix of transfer functions $\tilde{L}_{ij}(\omega)$.

Write a concise expression for $S(\omega)$ in terms of $R(\omega)$ and $\tilde{L}(\omega)$.

F. Now say we know that each of the noise inputs are independent, and have a flat power spectrum, specifically, that $S(\omega)=I$. Can we deduce the matrix $\tilde{L}(\omega)$ from the matrix $R(\omega)$? Why or why not?

Solutions

A and C (A is the same as C, but with $N=2$):

First calculate Fourier estimates for the response $R_i(t)$: $\tilde{r}_i(\omega)=\sum_{k=1}^{N} L_{ik}(\omega)\tilde{s}_k(\omega)$.

Then,

$$P_{R_i}(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle \tilde{r}_i(\omega)\overline{\tilde{r}_i(\omega)}\right\rangle=\lim_{T\to\infty}\frac{1}{T}\left\langle\left(\sum_{k=1}^{N}L_{ik}(\omega)\tilde{s}_k(\omega)\right)\overline{\left(\sum_{m=1}^{N}L_{im}(\omega)\tilde{s}_m(\omega)\right)}\right\rangle$$

$$=\lim_{T\to\infty}\frac{1}{T}\left\langle\sum_{k=1}^{N}\sum_{m=1}^{N}L_{ik}(\omega)\overline{L_{im}(\omega)}\tilde{s}_k(\omega)\overline{\tilde{s}_m(\omega)}\right\rangle ,$$

$$=\sum_{k=1}^{N}\sum_{m=1}^{N}L_{ik}(\omega)\overline{L_{im}(\omega)}\lim_{T\to\infty}\frac{1}{T}\left\langle\sum_{k=1}^{N}\sum_{m=1}^{N}\tilde{s}_k(\omega)\overline{\tilde{s}_m(\omega)}\right\rangle$$

where the last step follows because the L's are fixed values, unaffected by the limiting process.

So

$$P_{R_i}(\omega)=\sum_{k=1}^{N}\sum_{m=1}^{N}L_{ik}(\omega)\overline{L_{im}(\omega)}\lim_{T\to\infty}\frac{1}{T}\left\langle\sum_{k=1}^{N}\sum_{m=1}^{N}\tilde{s}_k(\omega)\overline{\tilde{s}_m(\omega)}\right\rangle$$

$$=\sum_{k=1}^{N}\sum_{m=1}^{N}L_{ik}(\omega)\overline{L_{im}(\omega)}P_{S_k,S_m}(\omega)$$

, where we have used the notational convention that $P_{S_k,S_m}(\omega)=P_{S_k}(\omega)$ when $k=m$ (this is consistent because the cross-spectrum of a signal with itself is its spectrum).

B and D.

$$P_{R_i,R_j}(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle \tilde{r}_i(\omega)\overline{\tilde{r}_j(\omega)}\right\rangle=\lim_{T\to\infty}\frac{1}{T}\left\langle\left(\sum_{k=1}^{N}L_{ik}(\omega)\tilde{s}_k(\omega)\right)\overline{\left(\sum_{m=1}^{N}L_{jm}(\omega)\tilde{s}_m(\omega)\right)}\right\rangle$$

$$=\lim_{T\to\infty}\frac{1}{T}\left\langle\sum_{k=1}^{N}\sum_{m=1}^{N}L_{ik}(\omega)\overline{L_{jm}(\omega)}\tilde{s}_k(\omega)\overline{\tilde{s}_m(\omega)}\right\rangle .$$

$$=\sum_{k=1}^{N}\sum_{m=1}^{N}L_{ik}(\omega)\overline{L_{jm}(\omega)}\lim_{T\to\infty}\frac{1}{T}\left\langle\sum_{k=1}^{N}\sum_{m=1}^{N}\tilde{s}_k(\omega)\overline{\tilde{s}_m(\omega)}\right\rangle$$

$$=\sum_{k=1}^{N}\sum_{m=1}^{N}L_{ik}(\omega)\overline{L_{jm}(\omega)}P_{S_kS_m}(\omega)$$

E. From D, $P_{R_i,R_j}(\omega)=\sum_{k=1}^{N}\sum_{m=1}^{N}L_{ik}(\omega)\overline{L_{jm}(\omega)}P_{S_kS_m}(\omega)=\sum_{k=1}^{N}\sum_{m=1}^{N}\left(\tilde{L}(\omega)\right)_{ik}\left(\tilde{L}(\omega)^*\right)_{mj}P_{S_kS_m}(\omega)$, because the adjoint is the complex conjugate of the transpose.

So

$\left(R(\omega)\right)_{ij}=\sum_{k=1}^{N}\sum_{m=1}^{N}\left(\tilde{L}(\omega)\right)_{ik}\left(\tilde{L}(\omega)^{*}\right)_{mj}\left(S(\omega)\right)_{km}$, and $R(\omega)=\tilde{L}(\omega)S(\omega)\left(\tilde{L}(\omega)\right)^{*}$.

F. From E, for independent, white-noise inputs of unit power, $S(\omega)=I$, and $R(\omega)=\tilde{L}(\omega)\left(\tilde{L}(\omega)\right)^{*}$.
One cannot deduce $\tilde{L}(\omega)$ from $R(\omega)$. The simplest reason is that even for N=1 (where there is no issue of channels mixing), this cannot be done. Here, $R(\omega)=\tilde{L}(\omega)\left(\tilde{L}(\omega)\right)^{*}=\left|\tilde{L}(\omega)\right|^{2}$. The reason that L cannot be determined is that there are non-trivial filters with $\left|\tilde{U}(\omega)\right|^{2}=1$: the pure delay, and also, the example of LSBB14155b, homework question 2. Replacing $\tilde{L}(\omega)$ by $\tilde{L}(\omega)\tilde{U}(\omega)$, i.e., convolving L with U, does not change $R(\omega)=\left|\tilde{L}(\omega)\right|^{2}=\left|\tilde{L}(\omega)\tilde{U}(\omega)\right|^{2}$.

For $N>1$, there are further reasons that L cannot be determined, related to the mixing of signals. If $U(\omega)$ is any unitary matrix, then (by definition) $U(\omega)\left(U(\omega)\right)^{*}=I$, so
$R(\omega)=\tilde{L}(\omega)\left(\tilde{L}(\omega)\right)^{*}=\tilde{L}(\omega)\tilde{U}(\omega)\left(\tilde{U}(\omega)\right)^{*}\left(\tilde{L}(\omega)\right)^{*}=\tilde{L}(\omega)\tilde{U}(\omega)\left(\tilde{L}(\omega)\tilde{U}(\omega)\right)^{*}$. That is, following L by a network that mixes the signals according to a unitary matrix does not change the observed cross-spectral matrix R.