Exam, 2014-2015

Do a total of 30 points (more if you want, of course). Show your work!

Q1: 10 points (4 parts, 2 points each for the first two parts, 3 points each for the second two parts)
Q2: 10 points (11 parts, 1 point each, max credit 10 points)
Q3: 10 points (5 parts, 2 points each)
Q4 :12 points (6 parts, 2 points each)

Q1: Constructing a group representation based on cosets.

Let G be a finite group, and *H* a subgroup of *G*. Recall that the coset *Hb* is the set of all of the elements *g* of *G* that can be written in the form $g = h \circ b$, for some element *h* in *H*, and that the cosets constitute a partition of *G* into disjoint subsets.

Now consider the free vector space *W* on the cosets: that is, *W* be the vector space of functions from the cosets *Hb* to $\mathbb{C}$. Since W is a free vector space, it has a natural inner product, $\langle x, y \rangle = \sum_{\text{cosets } Hb} x(Hb)\overline{y(Hb)}$

For any function *x* in *W* and any element *p* of *G*, we define Q_p, a member of $Hom(W,W)$, as follows: Q_p takes *x* (a function on the cosets) to the $Q_p(x)$ (another function on the cosets) whose value at the coset *Hb* is given by $\left(Q_p(x)\right)(Hb) = x(Hbp)$.

A. Show that this is a group representation.
B. What is its dimension (in terms of the sizes $\#(G)$ of *G*, and $\#(H)$ of *H*)?
C. Suppose further that *G* is commutative. What is the character of *Q*?
D. (No longer supposing that *G* is commutative). Under what circumstances is *Q* irreducible?

Question 2. Permutation matrices and eigendecompositions

Consider an $n \times n$ cyclic permutation matrix *M*, defined by $m_{j,j+1} = 1$ for $1 \le j \le (n-1)$, $m_{n,1} = 1$, and otherwise zero. Here, for $n = 5$: $M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$.

A. Is *M* unitary?
B. Is *M* self-adjoint?
C. What are the eigenvalues of *M*?
D. What are the eigenvectors of *M*?
E. Consider a matrix *C*, which is cyclic but with entries not necessarily drawn from $\{0,1\}$. (so it is not a permutation matrix). Specifically, $c_{j,k} = f_{k-j}$, where $k - j$ is interpreted mod *n*. For example, for $n = 5$,

$C = \begin{pmatrix} f_0 & f_1 & f_2 & f_3 & f_4 \\ f_4 & f_0 & f_1 & f_2 & f_3 \\ f_3 & f_4 & f_0 & f_1 & f_2 \\ f_2 & f_3 & f_4 & f_0 & f_1 \\ f_1 & f_2 & f_3 & f_4 & f_0 \end{pmatrix}$. Is *C* unitary?

F. Is *C* self-adjoint?
G. Does *C* commute with *M*?
H. What are the eigenvectors of *C*?
I. What are the eigenvalues of *C*?
J. What are the eigenvalues of this matrix, which is a permutation matrix but not cyclic?

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

K. What can you say about the eigenvalues of any permutation matrix of size $n \times n$?

Question 3. Transfer function and power spectrum in a simple physical system

Say an output, $x(t)$, is related to an input, $s(t)$ by $m\frac{d^2x}{dt^2}=s-kx-c\frac{dx}{dt}$. That is, the output x is the position of an object of mass *m* that is subject to a sum of three forces: the input signal s, a spring-like restoring force $-kx$, and a frictional force $-c\frac{dx}{dt}$.

A. Determine the transfer function that relates x to *s*.

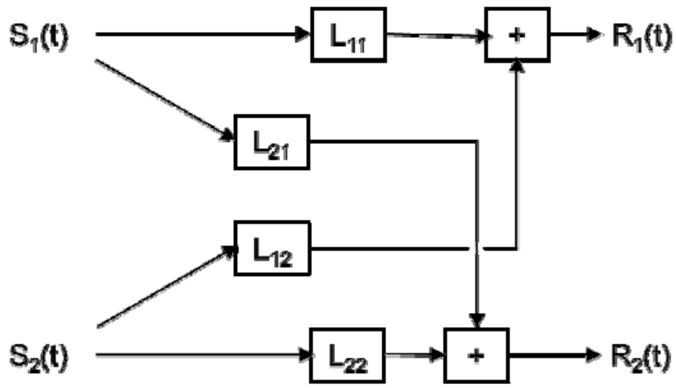
B. Say *s* is white noise. What is the power spectrum of *x*?

C and D. Under what circumstances does the power spectrum have a peak at a nonzero frequency? At what frequency is the peak?

E. Describe what happens if $c=0$ (no friction).

Question 4. Coherence and network identification

Say $S_1(t)$ and $S_2(t)$ are noise sources with power spectra $P_{S_1}(\omega)$ and $P_{S_2}(\omega)$, not necessarily independent, which are connected to two observable outputs $R_1(t)$ and $R_2(t)$ by the following network,where L_{ij} are linear filters with transfer functions $\tilde{L}_{ij}(\omega)$.



A. Find the power spectra $P_{R_1}(\omega)$ and $P_{R_2}(\omega)$ of the outputs in terms of the power spectra $P_{S_i}(\omega)$ and cross-spectrum $P_{S_1,S_2}(\omega)$ of the inputs.
B. Find the cross-spectrum of the outputs.
C and D. Same as A and B, but for *N* independent noise sources fully connected to *N* observable outputs, rather than two.
E. Define $S(\omega)$ as the "cross-spectral matrix" of *S*, i.e., the matrix whose elements are given by *S*
$S_{ij}(\omega) = \begin{cases} P_{S_i}(\omega), & i = j \\ P_{S_i,S_j}(\omega), & i \neq j \end{cases}$, and similarly for *R*. Further define $\tilde{L}(\omega)$ as the matrix of transfer functions $\tilde{L}_{ij}(\omega)$.
Write a concise expression for $S(\omega)$ in terms of $R(\omega)$ and $\tilde{L}(\omega)$.
F. Now say we know that each of the noise inputs are independent, and have a flat power spectrum, specifically, that $S(\omega) = I$. Can we deduce the matrix $\tilde{L}(\omega)$ from the matrix $R(\omega)$? Why or why not?