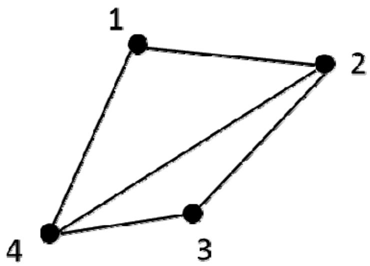
Graph-Theoretic Methods

Homework #1 (2016-2017), Answers

Q1: Group-theoretic eigen-decomposition of the Laplacian

Consider the following graph:



A. What is its adjacency matrix?

$$A=\begin{pmatrix}0&1&0&1\\1&0&1&1\\0&1&0&1\\1&1&1&0\end{pmatrix}.$$

B. What is its incidence matrix?

Many possibilities. Taking the edges (rows) in the order $1\sim 2$, $2\sim 3$, $3\sim 4$, $4\sim 1$, $2\sim 4$ and assigning $+1$ to the first of each: $Q=\begin{pmatrix}1&-1&0&0\\0&1&-1&0\\0&0&1&-1\\-1&0&0&1\\0&1&0&-1\end{pmatrix}$

C. What is its Laplacian?

$$L=D-A=\begin{pmatrix}2&0&0&0\\0&3&0&0\\0&0&2&0\\0&0&0&3\end{pmatrix}-\begin{pmatrix}0&1&0&1\\1&0&1&1\\0&1&0&1\\1&1&1&0\end{pmatrix}=\begin{pmatrix}2&-1&0&-1\\-1&3&-1&-1\\0&-1&2&-1\\-1&-1&-1&3\end{pmatrix}$$

or

$$L=Q^TQ=\begin{pmatrix}2&-1&0&-1\\-1&3&-1&-1\\0&-1&2&-1\\-1&-1&-1&3\end{pmatrix}$$

D. Note that the following four symmetry operations preserve the adjacency matrix (and also, the Laplacian): $G=\{e,(13),(24),(13)(24)\}$ *,where* e *is the identity,* (13) *denotes swapping the vertices* 1 *and* 3 *, etc. Show* G *is a group, and that it is a direct product of two subgroups :* $G_{13}=\{e,(13)\}$ *and* $G_{24}=\{e,(24)\}$.

To show G is a group: It's a set of permutations, so it is necessarily associative. Each element of G is its own inverse. So we just have to show closure, and this follows from the (trivial) $(13)\circ(24)=(13)(24)$. This last statement also suffices to show that $G=G_{13}\times G_{24}$.

E. Use $G=G_{13}\times G_{24}$ to find the complete set of irreducible representations of G.

Since G, G_{13}, and G_{24} are all commutative, all the irreducible representations are one-dimensional, and the number of such representations is equal to the size of each group.

So for G_{13}, there are two representations: the trivial one and one other. This can be found many ways – for example, it is the parity representation; for example it has a character that is the only orthonormal function to the character of the trivial representation. So the character table for G_{13} is:

	e	(13)
$\chi_{I_{13}}$	1	1
$\chi_{P_{13}}$	1	-1

and the character table for G_{24} is

	e	(24)
$\chi_{I_{24}}$	1	1
$\chi_{P_{24}}$	1	-1

.

The representations of $G=G_{13}\times G_{24}$ are now found by multiplying the characters of the components:

	e	$(13)\times e$	$e\times(24)$	$(13)\times(24)$
$\chi_{I_{13}\times I_{24}}$	1	1	1	1
$\chi_{P_{13}\times I_{24}}$	1	-1	1	-1
$\chi_{I_{13}\times P_{24}}$	1	1	-1	-1
$\chi_{P_{13}\times P_{24}}$	1	-1	-1	1

F. Now consider the permutation representation U corresponding to how G acts on functions on the graph. Find its character. Determine how many copies of each of the above irreducible representations ($I_{13}\times I_{24}, P_{13}\times I_{24}, I_{13}\times P_{24}, P_{13}\times P_{24}$) it contains.

Since it is a permutation representation – with all matrices containing just 0's and 1's -- the character is the number of vertices not moved by each group element.

	e	(13)	(24)	$(13)(24)$
χ_U	4	2	2	0

To determine the number of copies in U of each of the irreducible representations, we use the trace formula of the group representation theorem, $d(U,M)=\frac{1}{|G|}\sum_g \overline{\chi_U(g)}\chi_M(g)$, where $|G|=4$ and M is one of the above four irreducible representations.

$$d(U,I_{13}\times I_{24})=\frac{1}{4}\left(4\cdot 1+2\cdot 1+2\cdot 1+0\cdot 1\right)=2$$

$$d(U,P_{13}\times I_{24})=\frac{1}{4}\big(4\cdot 1+2\cdot(-1)+2\cdot 1+0\cdot(-1)\big)=1$$
$$d(U,I_{13}\times P_{24})=\frac{1}{4}\big(4\cdot 1+2\cdot 1+2\cdot(-1)+0\cdot(-1)\big)=1$$
$$d(U,P_{13}\times P_{24})=\frac{1}{4}\big(4\cdot 1+2\cdot(-1)+2\cdot(-1)+0\cdot 1\big)=0\,.$$

G. For each of the irreducible representations that occur in U, find the corresponding subspace of functions on the graph in which the irreducible representation acts.

For the identity representation, $I_{13}\times I_{24}$, we need to project onto the subspace in which U acts like the identity. In general, the projection onto the subspace in which a representation acts like the identity is given by

$P_U(v)=\frac{1}{|G|}\sum_g U_g(v)$. So, for $v=\begin{pmatrix}v_1\\v_2\\v_3\\v_4\end{pmatrix}$,

$$P_U(v)=\frac{1}{4}\big(U_e(v)+U_{(13)}(v)+U_{(24)}(v)+U_{(13)(24)}(v)\big)$$
$$=\frac{1}{4}\left(\begin{pmatrix}v_1\\v_2\\v_3\\v_4\end{pmatrix}+\begin{pmatrix}v_3\\v_2\\v_1\\v_4\end{pmatrix}+\begin{pmatrix}v_1\\v_4\\v_3\\v_2\end{pmatrix}+\begin{pmatrix}v_3\\v_4\\v_1\\v_2\end{pmatrix}\right)=\begin{pmatrix}(v_1+v_3)/2\\(v_2+v_4)/2\\(v_1+v_3)/2\\(v_2+v_4)/2\end{pmatrix}.$$

So the subspace in which U acts like $I_{13}\times I_{24}$ is a two-dimensional, and consists of vectors of the form $\begin{pmatrix}x\\y\\x\\y\end{pmatrix}$.

In the subspace in which U acts like $P_{13}\times I_{24}$, we need a function on the vertices that is multiplied by -1 when applying the group operation (13). Since this swaps the vertices 1 and 3, the values assigned to vertices 1 and 3 must be opposite. But it also must negate the values assigned to verteices 2 and 4, even though (13) does not move these vertices. Therefore , it must assign the values on vertices 2 and 4 to zero.

So the subspace in which U acts like $P_{13}\times I_{24}$ is one -dimensional, and consists of vectors of the form $\begin{pmatrix}u\\0\\-u\\0\end{pmatrix}$.

Similarly, the U acts like $I_{13}\times P_{24}$ in the subspace consisting of vectors of the form $\begin{pmatrix}0\\v\\0\\-v\end{pmatrix}$.

H. Use the fact that the eigenvectors of the graph Laplacian must lie in the subspaces identified in part G to find its eigenvectors and eigenvalues.

Any vector in a one-dimensional subspace must be an eigenvector. For the subspace in which U acts like $P_{13} \times I_{24}$:

$$L\begin{pmatrix} u \\ 0 \\ -u \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}\begin{pmatrix} u \\ 0 \\ -u \\ 0 \end{pmatrix} = \begin{pmatrix} 2u \\ 0 \\ -2u \\ 0 \end{pmatrix} = 2\begin{pmatrix} u \\ 0 \\ -u \\ 0 \end{pmatrix}$$, so this is an eigenvector with eigenvalue 2.

For the subspace in which U acts like $I_{13} \times P_{24}$:

$$L\begin{pmatrix} 0 \\ v \\ 0 \\ -v \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ v \\ 0 \\ -v \end{pmatrix} = \begin{pmatrix} 0 \\ 4v \\ 0 \\ -4v \end{pmatrix} = 4\begin{pmatrix} 0 \\ v \\ 0 \\ -v \end{pmatrix}$$, so this is an eigenvector with eigenvalue 4.

For the two-dimesnsional subspace in which U acts like $I_{13} \times I_{24}$:, we have to work a little harder:

$$L\begin{pmatrix} x \\ y \\ x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ x \\ y \end{pmatrix} = \begin{pmatrix} 2x-2y \\ -2x+2y \\ 2x-2y \\ -2x+2y \end{pmatrix} = 2\begin{pmatrix} x-y \\ -x+y \\ x-y \\ -x+y \end{pmatrix}.$$

So, in the two-dimensional subspace, choosing coordinates $\begin{pmatrix} x \\ y \\ x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, the matrix representation of M is $M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. $\det M = 0$ and $trM = 2$ This is a matrix whose characteristic equation is $\lambda^2 - 2\lambda = 0$. Its eigenvalues are therefore 0 and 2. The zero eigenvalue corresponds to the uniform eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and the eigenvalue 2 corresponds to an eigenvector $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (it must be orthogonal), and hence, to $\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$ on the original graph.

So the graph Laplacian has the following four eigenvalues and corresponding eigenvectors:

In the subspace in which U acts like $I_{13} \times I_{24}$: $\lambda = 0$, $v = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $\lambda = 2$, $v = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$.

In the subspace in which U acts like $P_{13} \times I_{24}$: $\lambda = 2$ and $\lambda = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$.

In the subspace in which U acts like $I_{13} \times P_{24}$: $\lambda = 4$ and $\lambda = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$.

Q2: "Laplacians" of a directed graph

Consider a directed graph in which vertex 1 is connected to vertex 2, vertex 2 to vertex 3, and vertex 3 to vertex 1. Define the Laplacian as in the notes: $L = D - A$, but now the adjacency matrix A is not symmetric, and the degree matrix, D is the number of outgoing connections.

A. What is the graph Laplacian?

$$L = D - A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}.$$

B. What are its eigenvalues?

Eigenvalues are the roots of the characteristic equation $\det(\lambda I - L) = 0$.

$$\det(\lambda I - L) = \det \begin{pmatrix} \lambda - 1 & 1 & 0 \\ 0 & \lambda - 1 & 1 \\ 1 & 0 & \lambda - 1 \end{pmatrix} = (\lambda - 1)^3 + 1.$$

If $(\lambda - 1)^3 + 1 = 0$, then $\lambda - 1 = -\omega_k$, where the $\omega_k = e^{\frac{2\pi i}{3}k}$ are the three cube roots of unity, and $\lambda = 1 - \omega_k$.