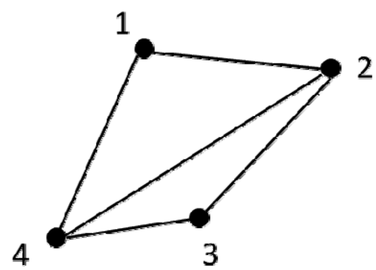
Graph-Theoretic Methods

Homework #1 (2016-2017), Questions

Q1: Group-theoretic eigen-decomposition of the Laplacian

Consider the following graph:



A. What is its adjacency matrix?

B. What is its incidence matrix?

C. What is its Laplacian?

D. Note that the following four symmetry operations preserve the adjacency matrix (and also, the Laplacian): $G = \{e,(13),(24),(13)(24)\}$,where e is the identity, (13) denotes swapping the vertices 1 and 3 , etc. Show G is a group, and that it is a direct product of two subgroups : $G_{13} = \{e,(13)\}$ and $G_{24} = \{e,(24)\}$.

E. Use $G = G_{13} \times G_{24}$ to find the complete set of irreducible representations of G .

F. Now consider the permutation representation U corresponding to how G acts on functions on the graph. Find its character. Determine how many copies of each of the above irreducible representations ($I_{13} \times I_{24}, P_{13} \times I_{24}, I_{13} \times P_{24}, P_{13} \times P_{24}$) it contains.

G. For each of the irreducible representations that occur in U , find the corresponding subspace of functions on the graph in which the irreducible representation acts.

H. Use the fact that the eigenvectors of the graph Laplacian must lie in the subspaces identified in part G to find its eigenvectors and eigenvalues.

Q2: “Laplacians” of a directed graph

Consider a directed graph in which vertex 1 is connected to vertex 2, vertex 2 to vertex 3, and vertex 3 to vertex 1. Define the Laplacian as in the notes: $L = D - A$, but now the adjacency matrix A is not symmetric, and the degree matrix, D is the number of outgoing connections.

A. What is the graph Laplacian?

B. What are its eigenvalues?