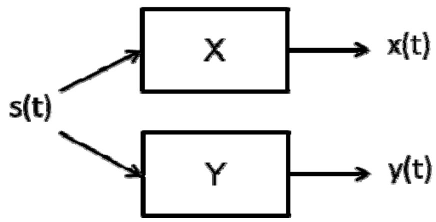
Linear Systems, Black Boxes, and Beyond

Homework #1 (2016-2017), Answers

Q1: Cross-spectra and coherences: one input in common and no added noise

Setup: one signal $s(t)$ *provides an input to two linear filters* X *and* Y*, with transfer functions* $\tilde{X}(\omega)$ *and* $\tilde{Y}(\omega)$*, and that* $x(t)$ *and* $y(t)$ *are their outputs. Assume* $s(t)$ *has the power spectrum* $P_S(\omega)$*.*



A) Calculate the power spectra $P_X(\omega)$ *of* $x(t)$*,* $P_Y(\omega)$ *of* $y(t)$*, and* $P_{X+Y}(\omega)$ *of their sum, and demonstrate that* $P_{X+Y}(\omega) \neq P_Y(\omega) + P_Y(\omega)$*.*

$P_X(\omega) = \lim_{T\to\infty} \frac{1}{T}\left\langle \tilde{x}(\omega)\overline{\tilde{x}(\omega)} \right\rangle$, and $\tilde{x}(\omega) = \tilde{X}(\omega)\tilde{s}(\omega)$, so

$P_X(\omega) = \lim_{T\to\infty} \frac{1}{T}\left\langle \tilde{X}(\omega)\tilde{s}(\omega)\overline{\tilde{X}(\omega)\tilde{s}(\omega)} \right\rangle = \tilde{X}(\omega)\overline{\tilde{X}(\omega)}\lim_{T\to\infty}\frac{1}{T}\left\langle \tilde{s}(\omega)\overline{\tilde{s}(\omega)}\right\rangle = \left|\tilde{X}(\omega)\right|^2 P_S(\omega)$.

Similarly, $P_Y(\omega) = \left|\tilde{Y}(\omega)\right|^2 P_S(\omega)$.

For the summed signal,

$$
\begin{aligned}
P_{X+Y}(\omega) &= \lim_{T\to\infty}\frac{1}{T}\left\langle \left(\tilde{X}(\omega)+\tilde{Y}(\omega)\right)\tilde{s}(\omega)\overline{\left(\tilde{X}(\omega)+\tilde{Y}(\omega)\right)\tilde{s}(\omega)}\right\rangle \\
&= \left(\tilde{X}(\omega)+\tilde{Y}(\omega)\right)\overline{\left(\tilde{X}(\omega)+\tilde{Y}(\omega)\right)}P_S(\omega) \\
&= \left(\tilde{X}(\omega)\overline{\tilde{X}(\omega)}+\tilde{X}(\omega)\overline{\tilde{Y}(\omega)}+\tilde{Y}(\omega)\overline{\tilde{X}(\omega)}+\tilde{Y}(\omega)\overline{\tilde{Y}(\omega)}\right)P_S(\omega) \\
&= \left(\left|\tilde{X}(\omega)\right|^2+\left|\tilde{Y}(\omega)\right|^2+\tilde{X}(\omega)\overline{\tilde{Y}(\omega)}+\tilde{Y}(\omega)\overline{\tilde{X}(\omega)}\right)P_S(\omega)
\end{aligned}.
$$

Combining the above, $P_{X+Y}(\omega) - P_X(\omega) - P_Y(\omega) = \left(\tilde{X}(\omega)\overline{\tilde{Y}(\omega)}+\tilde{Y}(\omega)\overline{\tilde{X}(\omega)}\right)P_S(\omega)$.

B) Define the cross-spectrum of $x(t)$ *and* $y(t)$ *as* $P_{X,Y}(\omega) = \lim_{T\to\infty}\frac{1}{T}\left\langle \tilde{x}(\omega)\overline{\tilde{y}(\omega)}\right\rangle$*. Write an expression for* $P_{X,Y}(\omega)$*,* $P_{X,S}(\omega)$*, and* $P_{Y,S}(\omega)$ *in terms of* $\tilde{X}(\omega)$*,* $\tilde{Y}(\omega)$*, and* $P_S(\omega)$*.*

Using $\tilde{x}(\omega) = \tilde{X}(\omega)\tilde{s}(\omega)$ and $\tilde{y}(\omega) = \tilde{Y}(\omega)\tilde{s}(\omega)$,

$$
\begin{aligned}
P_{X,Y}(\omega) &= \lim_{T\to\infty}\frac{1}{T}\left\langle \tilde{x}(\omega)\overline{\tilde{y}(\omega)}\right\rangle = \lim_{T\to\infty}\frac{1}{T}\left\langle \tilde{X}(\omega)\tilde{s}(\omega)\overline{\tilde{Y}(\omega)\tilde{s}(\omega)}\right\rangle \\
&= \tilde{X}(\omega)\overline{\tilde{Y}(\omega)}\lim_{T\to\infty}\frac{1}{T}\left\langle \tilde{s}(\omega)\overline{\tilde{s}(\omega)}\right\rangle = \tilde{X}(\omega)\overline{\tilde{Y}(\omega)}P_S(\omega)
\end{aligned}.
$$

Similarly,

$$P_{X,S}(\omega) = \lim_{T\to\infty} \frac{1}{T}\left\langle \tilde{x}(\omega)\overline{\tilde{s}(\omega)} \right\rangle = \lim_{T\to\infty} \frac{1}{T}\left\langle \tilde{X}(\omega)\tilde{s}(\omega)\overline{\tilde{s}(\omega)} \right\rangle = \tilde{X}(\omega)\lim_{T\to\infty} \frac{1}{T}\left\langle \tilde{s}(\omega)\overline{\tilde{s}(\omega)} \right\rangle = \tilde{X}(\omega)P_S(\omega)$$

and

$P_{Y,S}(\omega) = \tilde{Y}(\omega)P_S(\omega)$.

C) Write an expression for $P_{X+Y}(\omega)$ in terms of $P_{X,Y}(\omega)$, $P_X(\omega)$ and $P_Y(\omega)$.

From part A, $P_{X+Y}(\omega) = \left(\left|\tilde{X}(\omega)\right|^2 + \left|\tilde{Y}(\omega)\right|^2 + \tilde{X}(\omega)\overline{\tilde{Y}(\omega)} + \tilde{Y}(\omega)\overline{\tilde{X}(\omega)} \right) P_S(\omega)$. From Part B, $P_{X,Y}(\omega) = \tilde{X}(\omega)\overline{\tilde{Y}(\omega)}P_S(\omega)$. So $P_{X+Y}(\omega) = \left(\left|\tilde{X}(\omega)\right|^2 + \left|\tilde{Y}(\omega)\right|^2 \right) P_S(\omega) + P_{X,Y}(\omega) + \overline{P_{X,Y}(\omega)}$.

D) Define the coherence of $x(t)$ and $y(t)$ as $C_{X,Y}(\omega) = \frac{P_{X,Y}(\omega)}{\sqrt{P_X(\omega)P_Y(\omega)}}$. Find $C_{X,S}(\omega)$, $C_{Y,S}(\omega)$, and $C_{X,Y}(\omega)$.

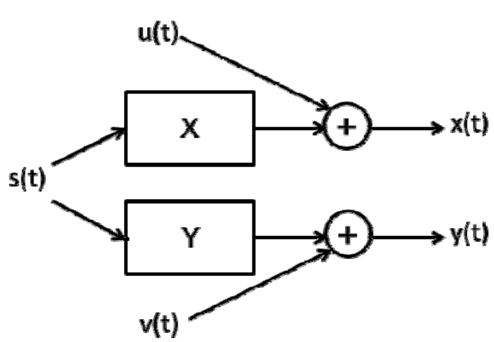
We start with $C_{X,S}(\omega) = \frac{P_{X,S}(\omega)}{\sqrt{P_X(\omega)P_S(\omega)}}$. Using $P_{X,S}(\omega) = \tilde{X}(\omega)P_S(\omega)$ from part B and $P_X(\omega) = \left|\tilde{X}(\omega)\right|^2 P_S(\omega)$ from part A, $C_{X,S}(\omega) = \frac{\tilde{X}(\omega)P_S(\omega)}{\sqrt{\left(\left|\tilde{X}(\omega)\right|^2 P_S(\omega)\right)P_S(\omega)}} = \frac{\tilde{X}(\omega)}{\sqrt{\left|\tilde{X}(\omega)\right|^2}} = \frac{\tilde{X}(\omega)}{\left|\tilde{X}(\omega)\right|}$, i.e., a complex number of magnitude 1 whose phase is the phase of $\tilde{X}(\omega)$. Similarly, $C_{Y,S}(\omega) = \frac{\tilde{Y}(\omega)}{\left|\tilde{Y}(\omega)\right|}$.

For $C_{X,Y}(\omega) = \frac{P_{X,Y}(\omega)}{\sqrt{P_X(\omega)P_Y(\omega)}}$, combining parts B and A yields

$$C_{X,Y}(\omega) = \frac{\tilde{X}(\omega)\overline{\tilde{Y}(\omega)}P_S(\omega)}{\sqrt{\left(\left|\tilde{X}(\omega)\right|^2 P_S(\omega)\right)\left(\left|\tilde{Y}(\omega)\right|^2 P_S(\omega)\right)}} = \frac{\tilde{X}(\omega)\overline{\tilde{Y}(\omega)}}{\left|\tilde{X}(\omega)\right|\left|\tilde{Y}(\omega)\right|}.$$

Q2: Cross-spectra and coherences: one input in common but also added noise

*Setup: one signal $s(t)$ provides an input to two linear filters X and Y, with transfer functions $\tilde{X}(\omega)$ and $\tilde{Y}(\omega)$. We observe $x(t)$ and $y(t)$, which are the outputs of these linear filters *after adding independent signals $u(t)$ and $v(t)$. Assume $s(t)$ has power spectrum $P_S(\omega)$, $u(t)$ has power spectrum $P_U(\omega)$, and $v(t)$ has power spectrum $P_V(\omega)$, and that $s(t)$, $u(t)$ and $v(t)$ are all independent.*



A) Calculate the power spectra $P_X(\omega)$ of $x(t)$ and $P_Y(\omega)$ of $y(t)$.

$P_X(\omega) = \lim_{T\to\infty} \frac{1}{T}\left\langle \tilde{x}(\omega)\overline{\tilde{x}(\omega)} \right\rangle$, and $\tilde{x}(\omega) = \tilde{X}(\omega)\tilde{s}(\omega) + \tilde{u}(\omega)$, so

$$P_X(\omega) = \lim_{T\to\infty} \frac{1}{T}\left\langle \left(\tilde{X}(\omega)\tilde{s}(\omega) + \tilde{u}(\omega)\right)\overline{\left(\tilde{X}(\omega)\tilde{s}(\omega) + \tilde{u}(\omega)\right)} \right\rangle$$
$$= \lim_{T\to\infty} \frac{1}{T}\left\langle \left(\tilde{X}(\omega)\tilde{s}(\omega)\right)\overline{\left(\tilde{X}(\omega)\tilde{s}(\omega)\right)} + \left(\tilde{X}(\omega)\tilde{s}(\omega)\right)\overline{\tilde{u}(\omega)} + \tilde{u}(\omega)\overline{\left(\tilde{X}(\omega)\tilde{s}(\omega)\right)} + \tilde{u}(\omega)\overline{\tilde{u}(\omega)} \right\rangle.$$

Since $s(t)$ *and* $u(t)$ *are independent, the cross-terms are zero.*

$$P_X(\omega) = \lim_{T\to\infty} \frac{1}{T}\left\langle \left(\tilde{X}(\omega)\tilde{s}(\omega)\right)\overline{\left(\tilde{X}(\omega)\tilde{s}(\omega)\right)} + \tilde{u}(\omega)\overline{\tilde{u}(\omega)} \right\rangle = \left|\tilde{X}(\omega)\right|^2 P_S(\omega) + P_U(\omega).$$

Similarly,

$$P_Y(\omega) = \left|\tilde{Y}(\omega)\right|^2 P_S(\omega) + P_V(\omega).$$

B) Calculate the cross spectrum of $x(t)$ *and* $y(t)$.

Using $\tilde{x}(\omega) = \tilde{X}(\omega)\tilde{s}(\omega) + \tilde{u}(\omega)$ and $\tilde{y}(\omega) = \tilde{Y}(\omega)\tilde{s}(\omega) + \tilde{v}(\omega)$:

$$P_{X,Y}(\omega) = \lim_{T\to\infty} \frac{1}{T}\left\langle \tilde{x}(\omega)\overline{\tilde{y}(\omega)} \right\rangle = \lim_{T\to\infty} \frac{1}{T}\left\langle \left(\tilde{X}(\omega)\tilde{s}(\omega) + \tilde{u}(\omega)\right)\overline{\left(\tilde{Y}(\omega)\tilde{s}(\omega) + \tilde{v}(\omega)\right)} \right\rangle$$
$$= \tilde{X}(\omega)\overline{\tilde{Y}(\omega)} \lim_{T\to\infty} \frac{1}{T}\left\langle \tilde{s}(\omega)\overline{\tilde{s}(\omega)} \right\rangle = \tilde{X}(\omega)\overline{\tilde{Y}(\omega)} P_S(\omega),$$

since, as in A, the cross-terms re zero.

C) Calculate the coherence of $x(t)$ *and* $y(t)$.

We start with the definition $C_{X,Y}(\omega) = \frac{P_{X,Y}(\omega)}{\sqrt{P_X(\omega)P_Y(\omega)}}$. Combining parts A and B yields

$$C_{X,Y}(\omega) = \frac{\tilde{X}(\omega)\overline{\tilde{Y}(\omega)} P_S(\omega)}{\sqrt{\left(\left|\tilde{X}(\omega)\right|^2 P_S(\omega) + P_U(\omega)\right)\left(\left|\tilde{Y}(\omega)\right|^2 P_S(\omega) + P_V(\omega)\right)}}$$
$$= \frac{\tilde{X}(\omega)\overline{\tilde{Y}(\omega)}}{\left(\left|\tilde{X}(\omega)\right| + \sqrt{\frac{P_U(\omega)}{P_S(\omega)}}\right)\left(\left|\tilde{Y}(\omega)\right| + \sqrt{\frac{P_V(\omega)}{P_S(\omega)}}\right)}.$$

Note that this is typically less than 1, because of the “noise terms” related to the unshared inputs, $u(t)$ and $v(t)$.