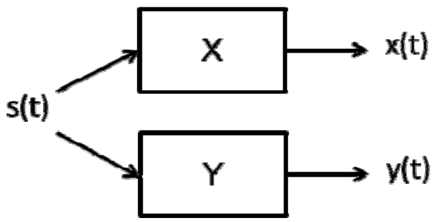
Linear Systems, Black Boxes, and Beyond

Homework #1 (2016-2017), Answers

Q1: Cross-spectra and coherences: one input in common and no added noise

Setup: one signal $s(t)$ provides an input to two linear filters X and Y, with transfer functions $\tilde{X}(\omega)$ and $\tilde{Y}(\omega)$, and that $x(t)$ and $y(t)$ are their outputs. Assume $s(t)$ has the power spectrum $P_S(\omega)$.



A) Calculate the power spectra $P_X(\omega)$ of $x(t)$, $P_Y(\omega)$ of $y(t)$, and $P_{X+Y}(\omega)$ of their sum, and demonstrate that $P_{X+Y}(\omega) \neq P_Y(\omega) + P_Y(\omega)$.

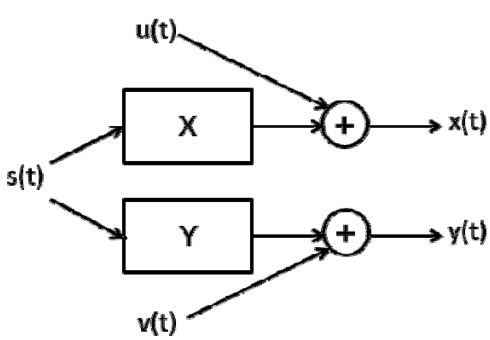
B) Define the cross-spectrum of $x(t)$ and $y(t)$ as $P_{X,Y}(\omega) = \lim_{T\to\infty} \frac{1}{T}\left\langle \tilde{x}(\omega)\overline{\tilde{y}(\omega)} \right\rangle$. Write an expression for $P_{X,Y}(\omega)$, $P_{X,S}(\omega)$, and $P_{Y,S}(\omega)$ in terms of $\tilde{X}(\omega)$, $\tilde{Y}(\omega)$, and $P_S(\omega)$.

C) Write an expression for $P_{X+Y}(\omega)$ in terms of $P_{X,Y}(\omega)$, $P_X(\omega)$ and $P_Y(\omega)$.

D) Define the coherence of $x(t)$ and $y(t)$ as $C_{X,Y}(\omega) = \frac{P_{X,Y}(\omega)}{\sqrt{P_X(\omega)P_Y(\omega)}}$. Find $C_{X,S}(\omega)$, $C_{Y,S}(\omega)$, and $C_{X,Y}(\omega)$.

Q2: Cross-spectra and coherences: one input in common but also added noise

*Setup: one signal $s(t)$ provides an input to two linear filters X and Y, with transfer functions $\tilde{X}(\omega)$ and $\tilde{Y}(\omega)$. We observe $x(t)$ and $y(t)$, which are the outputs of these linear filters *after adding independent signals $u(t)$ and $v(t)$. Assume $s(t)$ has power spectrum $P_S(\omega)$, $u(t)$ has power spectrum $P_U(\omega)$,and $v(t)$ has power spectrum $P_V(\omega)$, and that $s(t)$, $u(t)$ and $v(t)$ are all independent.*



A) Calculate the power spectra $P_X(\omega)$ of $x(t)$ and $P_Y(\omega)$ of $y(t)$.

B) Calculate the cross spectrum of $x(t)$ and $y(t)$.

C) Calculate the coherence of $x(t)$ and $y(t)$.