Linear Transformations and Group Representations

Homework #1 (2016-2017), Answers

*Q1: Another mapping from a group (the rotations of a circle) into linear operators. Here, $V$ is a two-dimensional vector space.*

*A. Find the eigenvalues of the transformation*

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Find the the characteristic equation:

$$\det(zI - R) = \det\begin{pmatrix} z-\cos\theta & -\sin\theta \\ \sin\theta & z-\cos\theta \end{pmatrix} = (z-\cos\theta)(z-\cos\theta) - (-\sin\theta)(\sin\theta)$$

$$= z^2 - 2z\cos\theta + \cos^2\theta + \sin^2\theta \quad .$$

$$= z^2 - 2z\cos\theta + 1$$

The eigenvalues are the roots of the characteristic equation, which we find by the quadratic formula:

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \cos\theta \pm \sqrt{\cos^2\theta - 1}$$

$$= \cos\theta \pm \sqrt{-\sin^2\theta} \quad .$$

$$= \cos\theta \pm i\sin\theta$$

$$= e^{\pm i\theta}$$

So there are two eigenvalues, $e^{i\theta}$ and $e^{-i\theta}$.

*B. Find its eigenvectors.*

We seek vectors $\vec{x} = \begin{pmatrix} u \\ v \end{pmatrix}$ for which $R\vec{x} = e^{i\theta}\vec{x}$ (and also, $R\vec{x} = e^{-i\theta}\vec{x}$). Looking just at the eigenvalue $e^{i\theta}$,

$R\vec{x} = e^{i\theta}\vec{x}$ implies $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = e^{i\theta}\begin{pmatrix} u \\ v \end{pmatrix}$, i.e.,

$u\cos\theta + v\sin\theta = e^{i\theta}u$ and $-u\sin\theta + v\cos\theta = e^{i\theta}v$. Using $e^{i\theta} = \cos\theta + i\sin\theta$, the first equation is equivalent to $u\cos\theta + v\sin\theta = u\cos\theta + iu\sin\theta$, i.e., $v = iu$, and the second equation is equivalent to $-u\sin\theta + v\cos\theta = v\cos\theta + iv\sin\theta$, i.e., $-u = iv$. So both equations solve with $v = iu$, i.e., the eigenvector is $\vec{x}_+ = \begin{pmatrix} 1 \\ i \end{pmatrix}$ (and any multiple of it).

Similarly, for the eigenvalue $e^{-i\theta}$, the eigenvector is $\vec{x}_- = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ (and any multiple of it).

*C. Since all of the transformations $R_\theta$ have the same eigenvectors (as shown in part B), they should commute. That is, $R_\theta R_\varphi = R_\varphi R_\theta$. Verify this.*

$$R_\theta R_\varphi = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\cos\varphi - \sin\theta\sin\varphi & \cos\theta\sin\varphi + \sin\theta\cos\varphi \\ -\sin\theta\cos\varphi - \cos\theta\sin\varphi & -\sin\theta\sin\varphi + \cos\theta\cos\varphi \end{pmatrix}.$$

$$= \begin{pmatrix} \cos(\theta+\varphi) & \sin(\theta+\varphi) \\ -\sin(\theta+\varphi) & \cos(\theta+\varphi) \end{pmatrix} = R_{\theta+\varphi}$$

Therefore, $R_\theta R_\varphi = R_{\theta+\varphi} = R_{\varphi+\theta} = R_\varphi R_\theta$, so we have shown that $R_\theta R_\varphi = R_\varphi R_\theta$.

*Q2: Eigenvalues and eigenvectors in a function space. Here, $V$ is the vector space of functions $f$ on the real line. Consider the mapping $H$, defined by $Hf(x) = \frac{d^2 f}{dx^2}(x) - x^2 f(x)$.*

*A. Show that $H$ is linear.*

We need to show that $H$ preserves addition and scalar multiplication. To show that $H$ preserves addition:

$$H(f+g)(x) = \frac{d^2}{dx^2}(f+g)(x) - x^2\left(f(x)+g(x)\right)$$

$$= \frac{d^2 f}{dx^2}(x) + \frac{d^2 g}{dx^2}(x) - x^2 f(x) - x^2 g(x)$$

$$= \frac{d^2 f}{dx^2}(x) - x^2 f(x) + \frac{d^2 g}{dx^2}(x) - x^2 g(x)$$

$$= H(f)(x) + H(g)(x)$$

To show that $H$ preserves scalar multiplication:

$$H(\alpha f)(x) = \frac{d^2}{dx^2}(\alpha f)(x) - x^2 \alpha f(x) = \alpha\left(\frac{d^2 f}{dx^2}(x) - x^2 f(x)\right) = \alpha H(f)(x).$$

*B. Show that $u_0(x) = e^{-x^2/2}$ is an eigenvector of $H$, and find its eigenvalue.*

If $u_0(x) = e^{-x^2/2}$, then $\frac{d}{dx}u_0(x) = -xe^{-x^2/2}$, and $\frac{d^2}{dx^2}u_0(x) = \frac{d}{dx}\left(-xe^{-x^2/2}\right) = x^2 e^{-x^2/2} - e^{-x^2/2}$. So $Hu_0(x) = \frac{d^2}{dx^2}u_0(x) - x^2 u_0(x) = -e^{-x^2/2} = -u_0(x)$, so the eigenvalue is $-1$.

*C. Show that $u_1(x) = xe^{-x^2/2}$ is an eigenvector of $H$, and find its eigenvalue.*

If $u_1(x) = xe^{-x^2/2}$, then $\frac{d}{dx}u_1(x) = -x^2e^{-x^2/2} + e^{-x^2/2}$, and

$\frac{d^2}{dx^2}u_1(x) = \frac{d}{dx}\left(-x^2e^{-x^2/2} + e^{-x^2/2}\right) = x^3e^{-x^2/2} - 2xe^{-x^2/2} - xe^{-x^2/2} = x^3e^{-x^2/2} - 3xe^{-x^2/2}$. So

$Hu_1(x) = \frac{d^2}{dx^2}u_1(x) - x^2u_1(x) = -3xe^{-x^2/2} = -3u_1(x)$, so the eigenvalue is $-3$.

Q3. Eigenvalues of a permutation matrix. Say $M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, so

$$M\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ c \\ a \end{pmatrix}.$$

*A. Show that* $M^3 = I$.

$M^3\begin{pmatrix} a \\ b \\ c \end{pmatrix} = M^2M\begin{pmatrix} a \\ b \\ c \end{pmatrix} = M^2\begin{pmatrix} b \\ c \\ a \end{pmatrix} = MM\begin{pmatrix} b \\ c \\ a \end{pmatrix} = M\begin{pmatrix} c \\ a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. (Or, we could just compute $M^3$ by matrix multiplication.)

*B. What are the eigenvalues of* $M$ *?*

Say $v$ is an eigenvector and $\lambda$ is its eigenvalue. Then $M^3v = v$, since $M^3$ is the identity (by part A).But also, $M^3v = M^2(Mv) = M^2\lambda v = \lambda M^2v = (\lambda M)(Mv) = (\lambda M)(\lambda v) = \lambda^2Mv = \lambda^3v$. So $\lambda^3v = v$, and $\lambda^3 = 1$.

So the three eigenvalues are the roots of $\lambda^3 = 1$, namely, 1, $e^{\frac{2\pi i}{3}}$ and $e^{\frac{4\pi i}{3}}$.