Linear Transformations and Group Representations

Homework #1 (2016-2017), Questions

Q1: Another mapping from a group (the rotations of a circle) into linear operators. Here, V is a two-dimensional vector space.

A. Find the eigenvalues of the transformation

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

B. Find its eigenvectors.

C. Since all of the transformations R_θ have the same eigenvectors (as shown in part B), they should commute. That is, $R_\theta R_\varphi = R_\varphi R_\theta$. Verify this.

Q2: Eigenvalues and eigenvectors in a function space. Here, V is the vector space of functions f on the real line. Consider the mapping H, defined by $Hf(x) = \frac{d^2 f}{dx^2}(x) - x^2 f(x)$.

A. Show that H is linear.

B. Show that $u_0(x) = e^{-x^2/2}$ is an eigenvector of H, and find its eigenvalue.

C. Show that $u_1(x) = xe^{-x^2/2}$ is an eigenvector of H, and find its eigenvalue.

Q3. Eigenvalues of a permutation matrix. Say $M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, so

$$M\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ c \\ a \end{pmatrix}.$$

A. Show that $M^3 = I$.

B. What are the eigenvalues of M ?