Linear Transformations and Group Representations

Homework #2 (2016-2017), Answers

Q1: Let M be the matrix representation of a permutation. (By a "matrix representation of a permutation, we mean, for example, that $M=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$ *represents the permutation* $\begin{pmatrix}a\\b\\c\end{pmatrix}\to\begin{pmatrix}b\\c\\a\end{pmatrix}$ *since* $M\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}b\\c\\a\end{pmatrix}$*.) Show that M is unitary.*

First solution. We are dealing with explicit matrix representations, so we need to work in coordinates. Say that $\vec{v}=\begin{pmatrix}v_1\\\vdots\\v_N\end{pmatrix}$ and the permutation takes element j to element $\sigma(j)$, i.e.,

$$M\vec{v}=\begin{pmatrix}v_{\sigma(1)}\\\vdots\\v_{\sigma(N)}\end{pmatrix}.$$

We first observe that to show that M is unitary, it suffices to show that $\langle\vec{v},\vec{x}\rangle=\langle M\vec{v},M\vec{x}\rangle$, since, with $\vec{x}=M^{-1}\vec{w}$, this is the same as $\langle\vec{v},M^{-1}\vec{w}\rangle=\langle M\vec{v},MM^{-1}\vec{w}\rangle=\langle M\vec{v},\vec{w}\rangle$. Now note that $\langle\vec{v},\vec{x}\rangle=\sum_{j=1}^{N}v_jx_j$ while $\langle M\vec{v},M\vec{x}\rangle=\sum_{j=1}^{N}v_{\sigma(j)}x_{\sigma(j)}$; these sums are identical other than the order of the terms.

Second solution. In the above setup, $\langle M\vec{v},\vec{w}\rangle=\sum_{j=1}^{N}v_{\sigma(j)}w_j$. Since the mapping $j\to\sigma(j)$ is a permutation, as j ranges over $1,\ldots,N$, then so does $\sigma(j)$. We can then reorder the above sum so that it is carried out in the order determined by $k=\sigma(j)$. That is,

$\langle M\vec{v},\vec{w}\rangle=\sum_{j=1}^{N}v_{\sigma(j)}w_j=\sum_{k=1}^{N}v_kw_{\sigma^{-1}(k)}$. We then need to show that $M^{-1}\vec{w}=\begin{pmatrix}w_{\sigma^{-1}(1)}\\\vdots\\w_{\sigma^{-1}(N)}\end{pmatrix}$, i.e., that the matrix corresponding to the permutation σ^{-1} is the inverse of the matrix corresponding to the permutation σ. This follows, because $MM^{-1}\vec{w}=M\begin{pmatrix}w_{\sigma^{-1}(1)}\\\vdots\\w_{\sigma^{-1}(N)}\end{pmatrix}=\begin{pmatrix}w_{\sigma(\sigma^{-1}(1))}\\\vdots\\w_{\sigma(\sigma^{-1}(N))}\end{pmatrix}=\begin{pmatrix}w_1\\\vdots\\w_N\end{pmatrix}=\vec{w}$. Finally,

$$\langle M\vec{v},\vec{w}\rangle = \sum_{j=1}^{N} v_{\sigma(j)} w_j = \sum_{k=1}^{N} v_k w_{\sigma^{-1}(k)} = \langle \vec{v}, M^{-1}\vec{w}\rangle .$$

Q2. Consider the Hilbert space of differentiable functions on the line for which $\int_{-\infty}^{\infty} |f(x)|^2\, dx$ *is finite, and with the inner product* $\langle f,g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$ *. Show that the linear operator defined by* $Lf(x) = i\dfrac{df}{dx}$ *is self-adjoint.*

$\langle Lf,g\rangle = \int_{-\infty}^{\infty} i\left(\frac{d}{dx} f(x)\right)\overline{g(x)}dx = if(x)g(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} if(x)\left(\frac{d}{dx}\overline{g(x)}\right)dx$ (the first equality is the definition of L, the second is integration by parts). Now note that if f and g have finite integrals, then they must go to zero for large values of x. So,

$$\langle Lf,g\rangle = -\int_{-\infty}^{\infty} if(x)\left(\frac{d}{dx}\overline{g(x)}\right)dx = \int_{-\infty}^{\infty} f(x)\left(\bar{i}\frac{d}{dx}\overline{g(x)}\right)dx = \int_{-\infty}^{\infty} f(x)\overline{(Lg)(x)}dx = \langle f,Lg\rangle .$$

Q3. Recall that a projection operator is a self-adjoint operator P *for which* $P^2 = P$ *.*

A. Show that if U *is unitary with* $U^N = I$ *, then* $Q = \dfrac{1}{N}\sum_{k=0}^{N-1} U^k$ *is a projection.*

First, show that Q is self-adjoint.

$\langle Qx,y\rangle = \left\langle \frac{1}{N}\sum_{k=0}^{N-1} U^k x, y\right\rangle = \frac{1}{N}\sum_{k=0}^{N-1}\langle U^k x,y\rangle = \frac{1}{N}\sum_{k=0}^{N-1}\left\langle x,\left(U^{-1}\right)^k y\right\rangle = \left\langle x, \frac{1}{N}\sum_{k=0}^{N-1}\left(U^{-1}\right)^k y\right\rangle$; the first equality is the definition of Q; the second is the linearity of the inner product; the third follows because U is unitary; the fourth from linearity. Now note that $\left(U^{-1}\right)^k = \left(U^k\right)^{-1} = U^{n-k}$, since $\left(U^k\right)U^{n-k} = U^n = 1$. So, $\langle Qx,y\rangle = \left\langle x, \frac{1}{N}\sum_{k=0}^{N-1} U^{N-k} y\right\rangle$. Now, note that as k runs from 0 to $N-1$, thenthe exponents of U, $N-k$, run from N to 1. Since $U^N = I = U^0$, this is the same as running from $N-1$ down to 0. So $\sum_{k=0}^{N-1} U^{N-k} = \sum_{k=0}^{N-1} U^N$, and $\langle Qx,y\rangle = \langle x,Qy\rangle$ as required.

Next, show that $Q^2 = Q$.

$Q^2 = \left(\frac{1}{N}\sum_{l=0}^{N-1} U^l\right)\left(\frac{1}{N}\sum_{m=0}^{N-1} U^m\right) = \frac{1}{N^2}\left(\sum_{l=0}^{N-1} U^l\right)\left(\sum_{m=0}^{N-1} U^m\right) = \frac{1}{N^2}\sum_{l=0}^{N-1}\sum_{m=0}^{N-1} U^{l+m}$. Note that, since $U^N = I$, $U^{l+m} = U^{l+m-N}$, so the final exponent can always be reduced (mod N) to an integer k ranging from 0 to $N-1$. So to simplify this sum, we need to count how many combinations of l and m result in a value of $l+m$ that is equal to k (mod N). However, $k = l+m$ (mod N)

is equivalent to $m = k - l$ (mod N), which means that for any value of k and l, there is always exactly one solution m in the range from 0 to $N-1$. So each value of the exponent $k = l + m$ can be achieved by exactly N pairs of values of l and m. So

$$Q^2 = \frac{1}{N^2}\sum_{l=0}^{N-1}\sum_{m=0}^{N-1}U^{l+m} = \frac{1}{N^2}\left(\sum_{k=0}^{N-1}NU^k\right) = \frac{1}{N}\left(\sum_{k=0}^{N-1}U^k\right) = Q$$, as required.

B. Let U be given by the permutation matrix corresponding to $\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \to \begin{pmatrix} b \\ c \\ a \\ d \\ f \\ e \end{pmatrix}$. *Compute the Q defined in part A, and also* $Q\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}$, *which directly verifies that Q is a projection.*

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.\ U^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.U^3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

So in U^k, the first three rows and columns become the identity for $k = 3, 6, \ldots$. The (4,4) element is always the identity. Row-and-columns 5 and 6 become the identity for $k = 2, 4, 6, \ldots$. So for $U^k = I$, k must be a multiple of 3 and of 2, i.e., $N = 6$.

$$Q = \frac{1}{6}\sum_{k=0}^{5}U^k = \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

Then $Q\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} \frac{a+b+c}{3} \\ \frac{a+b+c}{3} \\ \frac{a+b+c}{3} \\ d \\ \frac{e+f}{2} \\ \frac{e+f}{2} \end{pmatrix}$, i.e., Q averages the first three elements and the last two elements. So it is obviously a projection; averaging a second time does not change the values.