Linear Transformations and Group Representations

Homework #2 (2016-2017), Questions

Q1: Let M be the matrix representation of a permutation. (By a "matrix representation of a permutation, we mean, for example, that $M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ represents the permutation $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \to \begin{pmatrix} b \\ c \\ a \end{pmatrix}$ since $M \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ c \\ a \end{pmatrix}$.) Show that M is unitary.

Q2. Consider the Hilbert space of differentiable functions on the line for which $\int\limits_{-\infty}^{\infty} |f(x)|^2 \, dx$ is finite, and with the inner product $\langle f, g \rangle = \int\limits_{-\infty}^{\infty} f(x)\overline{g(x)}dx$. Show that the linear operator defined by $Lf(x) = i\dfrac{df}{dx}$ is self-adjoint.

Q3. Recall that a projection operator is a self-adjoint operator P for which $P^2 = P$.

A. Show that if U is unitary with $U^N = I$, then $Q = \dfrac{1}{N}\sum_{k=0}^{N-1} U^k$ is a projection.

B. Let U be given by the permutation matrix corresponding to $\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \to \begin{pmatrix} b \\ c \\ a \\ d \\ f \\ e \end{pmatrix}$. Compute the Q defined in part A, and also $Q\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}$, which directly verifies that Q is a projection.