Linear Transformations and Group Representations

Homework #3 (2016-2017), Answers

Q1-Q2 are further exercises concerning adjoints, self-adjoint transformations, and unitary transformations. Q3-6 involve group representations. Of these, Q3 and Q4 should be quick. Q5 is especially useful for the upcoming material.

Q1: (May be skipped, and result assumed for Q2): Let L be a linear transformation, and x and y scalars.Define e^{xL} by the power series $e^{xL}=\sum_{m=0}^{\infty}\frac{1}{m!}x^m L^m$. Show that $e^{xL}e^{yL}=e^{(x+y)L}$.

$$e^{xL}e^{yL}=\left(\sum_{m=0}^{\infty}\frac{1}{m!}x^m L^m\right)\left(\sum_{n=0}^{\infty}\frac{1}{n!}y^n L^n\right)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m!}\frac{1}{n!}x^m L^m y^n L^n=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m!}\frac{1}{n!}x^m y^n L^{m+n}.$$

Now collect terms with the same power of L, namely, $r=m+n$. For any r, there is a contribution with all m in the range $0,\ldots,r$, taking $n=r-m$. So,

$$e^{xL}e^{yL}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m!n!}x^m y^n L^{m+n}=\sum_{r=0}^{\infty}L^r\left(\sum_{m=0}^{r}\frac{1}{m!(r-m)!}x^m y^{r-m}\right)=\sum_{r=0}^{\infty}\frac{1}{r!}L^r\left(\sum_{m=0}^{r}\frac{r!}{m!(r-m)!}x^m y^{r-m}\right).$$

The inner sum in the last term simplifies because it is a binomial expansion of $(x+y)^r$:

$$e^{xL}e^{yL}=\sum_{r=0}^{\infty}\frac{1}{r!}L^r\left(\sum_{m=0}^{r}\frac{r!}{m!(r-m)!}x^m y^{r-m}\right)=\sum_{r=0}^{\infty}\frac{1}{r!}L^r(x+y)^r=\sum_{r=0}^{\infty}\frac{1}{r!}(x+y)^r L^r.$$

The final expression is the definition of $e^{(x+y)L}$.

Q2.A. For a general linear transformation A with adjoint A^* and (possibly complex) scalar z, show that the adjoint of zA is $\bar{z}A$.

$\langle zAv,w\rangle=z\langle Av,w\rangle=z\langle v,A^*w\rangle=\langle v,\bar{z}A^*w\rangle$, where the steps follow from (i) linearity of the inner product in the first argument, (ii) definition of the adjoint of A, (iii) conjugate linearity of the adjoint in the second argument.

B. Show that if A is self-adjoint and x is real (i.e, $x=\bar{x}$), then e^{ixA} is unitary, where e^{ixA} is defined as in Q1. Hint: do this by computing the adjoint of e^{ixA}.

$$\left\langle e^{ixA}\vec{v},\vec{w}\right\rangle=\left\langle\sum_{m=0}^{\infty}\frac{1}{m!}(ix)^m A^m\vec{v},\vec{w}\right\rangle=\sum_{m=0}^{\infty}\frac{1}{m!}(ix)^m\left\langle A^m\vec{v},\vec{w}\right\rangle=\sum_{m=0}^{\infty}\frac{1}{m!}(ix)^m\left\langle\vec{v},A^m\vec{w}\right\rangle,$$

where the equalities follow from (i) the definition of e^{ixA} as in Q1, (ii) linearity of the inner product in the first argument, and (iii) the assumption that A is self-adjoint. Now reverse the steps:

$\sum_{m=0}^{\infty}\frac{1}{m!}(ix)^m\left\langle\vec{v},A^m\vec{w}\right\rangle=\sum_{m=0}^{\infty}\left\langle\vec{v},\frac{1}{m!}(-ix)^m A^m\vec{w}\right\rangle=\left\langle\vec{v},\sum_{m=0}^{\infty}\frac{1}{m!}(-ix)^m A^m\vec{w}\right\rangle=\left\langle\vec{v},e^{-ixA}\vec{w}\right\rangle$ (first and second equality from conjugate linearity of the inner product in its second argument, third from the definition of e^{-ixA}. So the adjoint of e^{ixA} is e^{-ixA}. Finally, Q1 shows that these are inverses, since $e^{ixA}e^{-ixA} = e^{(ix-ix)A} = e^{0A} = I$. So the adjoint of e^{ixA} is its inverse, and it is therefore unitary.

C. An interesting special case. Recall that in a previous homework, we showed that the linear operator L, defined by $Lf(x) = i\dfrac{df}{dx}$ is self-adjoint in the Hilbert space of differentiable functions on the line for which $\int_{-\infty}^{\infty} |f(x)|^2\, dx$ is finite, and the inner product ia $\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$. Now we further assume that all derivatives of f exist. Apply the results of part B to $-sL$ (for s a real scalar) and develop a more familiar expression for the resulting operator.

According to part B, e^{-isL} is unitary, where . $-isLf(x) = (-is)i\dfrac{df}{dx} = s\dfrac{df}{dx}$. Applying the definition of an exponential of an operator in Q1:

$$e^{-isL} f(x) = e^{s\frac{d}{dx}} f(x) = \sum_{m=0}^{\infty} \frac{1}{m!} s^m \frac{d^m}{dx^m} f(x).$$

This is the familiar Taylor expansion for f around x. So if f is analytic (i.e., equal to its Taylor expansion, as "typical" functions are), then

$$e^{-isL} f(x) = \sum_{m=0}^{\infty} \frac{s^m}{m!} \frac{d^m}{dx^m} f(x) = f(x+s).$$

That is, $e^{-isL} = e^{s\frac{d}{dx}}$ is a shift by an amount *s*.

Q3: Let $g \to U_g$ be a unitary representation of a group G .Show that $g \to \det\left(U_g\right)$ is also a unitary representation, in a vector space of dimension 1.

First, we need to show that the operator $\det\left(U_g\right)$, i.e., multiplication by the scalar $\det\left(U_g\right)$, is unitary. This follows because (i) U_g is unitary so its eigenvalues all have magnitude 1; $\det\left(U_g\right)$ is the product of the eigenvalues of U_g, so it is multiplication by a scalar of magnitude 1.

Second, we need to show that the mapping $g \to \det\left(U_g\right)$ preserves structure, i.e., that $\det\left(U_{gh}\right) = \det\left(U_g\right)\det\left(U_h\right)$. This follows because U_g preserves structure, and the properties of the determinant. Since $U_{gh} = U_g U_h$, $\det(U_{gh}) = \det(U_g U_h) = \det(U_g)\det(U_h)$
.

Q4: Let $g \to U_g$ be a unitary representation of a group G .Is $g \to \left(U_g\right)^{-1}$ a unitary representation?

No, because in general, it fails to preserve the group structure: the inverse of a product is the product of the inverses, but in reverse order. $\left(U_{gh}\right)^{-1} = U_h^{-1} U_g^{-1}$, but for $g \to \left(U_g\right)^{-1}$ to be a representation, we'd need $\left(U_{gh}\right)^{-1}=U_g^{-1}U_h^{-1}$. If G is commutative, then this construction would have worked.

Q5. Character tables. Consider the group of rotations and mirror-flips of an equilateral triangle. Specifically, designate the three vertices as a, b, and c (in clockwise order, with a at the top), and the group operations as I for the identity, R and L for rotation right and left by 1/3 of a cycle, and M_a, M_b, and M_c for mirror flips along the lines through each of the vertices. Compute the characters at each of these elements for the representations described in the table below. With regard to S, recall (from earlier weeks) that a permutation is "odd" if it can be generated by an odd number of pair-swaps, and even if it requires an even number of pair swaps.

Group element:	I	R	L	M_a	M_b	M_c
Representation:						
E*: the trivial representation (all group elements map to 1)*						
P*: Representation as permutation matrices on the letters* $\{a,b,c\}$						
S*: Representation that maps even permutations on* $\{a,b,c\}$ *to* $+1$*, odd permutations to* -1						
C*: Representation as* 2×2 *change-of-coordinate matrices in the plane*						

The completed table follows this analysis:

E: In the trivial representation all group elements map to 1, so the character (the trace) is 1.

P: Each group element is mapped to a 3×3 permutation matrix. Its trace is the number of $1's$ on the diagonal, which is the number of letters in $\{a,b,c\}$ that are preserved. All three are preserved for the identity. For rotations, none are preserved, so the trace is 0. For mirror-flips, the vertex that is on the mirror line is preserved; the others are swapped, so the trace is 1.

S: A rotation is a cyclic permutation $(a,b,c)\to(b,c,a)$, and can be built by combining two pair-swaps, e.g., $(a,b)\to(b,a)$ and then $(a,c)\to(c,a)$, so it is even. A mirror flip preserves one vertex and swaps the other two, so it is odd.

C: The identity maps to the 2×2 identity matrix, whose trace is 2. A rotation by an angle θ maps to the matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, whose trace is $2\cos\theta$. For $\theta = \pm\frac{2\pi}{3}$, this is -1. The mirror-flip M_a is a flip along the vertical, so its matrix is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which has a trace of 0. M_b and M_c differ from M_a by a change of coordinates, so they must have the same trace.

Summarizing:

Group element:	I	R	L	M_a	M_b	M_c
Representation:						
E (trivial)	1	1	1	1	1	1
P (the letters $\{a,b,c\}$)	3	0	0	1	1	1
S (even and odd perms)	1	1	1	-1	-1	-1
C (2×2 matrices)	2	-1	-1	0	0	0

Note that $\chi_P = \chi_E + \chi_C$, suggesting (but not proving) that the representation P can be reduced into a direct sum of E and C.

Q6. Character of symmetric and antisymmetric parts of a tensor product of group representations. We start with the standard setup (in the class notes) for a tensor product of group representations: U_1 a representation of G in V_1, U_2 a representation of G in V_2, leading to a group representation $U_1 \otimes U_2$ in $V_1 \otimes V_2$ whose action is defined by $\left(U_{1,g} \otimes U_{2,g}\right)(v_1 \otimes v_2) = U_{1,g}(v_1) \otimes U_{2,g}(v_2)$. Here we add to this the further supposition that $U_1 = U_2 = U$, and $V_1 = V_2 = V$. Under these circumstances, recall (see notes concerning the derivation of the determinant) that $V \otimes V$ can be decomposed into two parts: a symmetric part $sym(V^{\otimes 2})$ which has a basis consisting of elements $v_i \otimes v_i$ and $\frac{1}{2}\left(v_i \otimes v_j + v_j \otimes v_i\right)$ (for $i < j$, and v_i a basis of V), and an antisymmetric part, $anti(V^{\otimes m})$, which has a basis consisting of elements $\frac{1}{2}\left(v_i \otimes v_j - v_j \otimes v_i\right)$ (for $i < j$).

A. Show that $U_g \otimes U_g$ maps $sym(V^{\otimes 2})$ into itself and also maps $anti(V^{\otimes m})$ into itself. So $U \otimes U = U^{\otimes 2}$ can be reduced into two components, $sym(U^{\otimes 2})$ and $anti(U^{\otimes 2})$. Here it is helpful to use a coordinate-free approach, where $sym(V^{\otimes 2})$ is the range of the projection P defined by $P(v \otimes v') = \frac{1}{2}\left(v \otimes v' + v' \otimes v\right)$, and $anti(V^{\otimes 2})$ is the range of the complementary projection $I - P$.

We need to show that $U_g \otimes U_g$ applied to an element in the range of P remains in the range of P. This will follow if we can show that $\left(U_g \otimes U_g\right)P = P\left(U_g \otimes U_g\right)$. To show the latter:

$$\left(U_g \otimes U_g\right)P(v \otimes v') = \left(U_g \otimes U_g\right)\left(\frac{1}{2}\left(v \otimes v' + v' \otimes v\right)\right) = \frac{1}{2}\left(\left(U_g \otimes U_g\right)(v \otimes v') + \left(U_g \otimes U_g\right)(v' \otimes v)\right)$$
$$= \frac{1}{2}\left(U_g(v) \otimes U_g(v') + U_g(v') \otimes U_g(v)\right) = P\left(U_g(v) \otimes U_g(v')\right) = P\left(U_g \otimes U_g\right)(v \otimes v').$$

Since this holds for all v and v', then $\left(U_g \otimes U_g\right)P = P\left(U_g \otimes U_g\right)$.

A similar argument can be used to show that $U_g \otimes U_g$ and $I - P$ commute. But it is easier not to c carry ouit another calculation ,but to observe that if any transformations Y and Z commute, then so do Y and $I - Z$, since I commutes with everything:

$$Y(I - Z) = YI - YZ = IY - ZY = (I - Z)Y.$$

B. Determine the characters of these two component representations ($sym(U^{\otimes 2})$ $and\, anti(U^{\otimes 2})$), in terms of the character of U. Here, to calculate $\chi_{sym(U^{\otimes 2})}(g) = tr(sym(U_g{}^{\otimes 2}))$ it is helpful use the bases $\frac{1}{2}\left(v_i \otimes v_j + v_j \otimes v_i\right)$ (for $i \le j$) for $sym(V^{\otimes 2})$, and to calculate $\chi_{anti(U^{\otimes 2})}(g) = tr(anti(U_g{}^{\otimes 2}))$, where the v_k are the eigenvectors of U_g. Similarly, it is helpful to use the basis $\frac{1}{2}\left(v_i \otimes v_j - v_j \otimes v_i\right)$ (for $i < j$) for $anti(V^{\otimes m})$.

For $sym(U^{\otimes 2})$: The trace is the sum of the eigenvalues, which we compute one eigenvector at a time. For the eigenvector $\frac{1}{2}\left(v_i \otimes v_j + v_j \otimes v_i\right)$, with $U_g(v_k) = \lambda_k \nu_k$,

$$sym(U_g^{\otimes 2})\left(\frac{1}{2}\left(v_i \otimes v_j + v_j \otimes v_i\right)\right) = \frac{1}{2}\left(sym(U_g^{\otimes 2})(v_i \otimes v_j) + sym(U_g^{\otimes 2})(v_j \otimes v_i)\right)$$
$$= \frac{1}{2}\left[\frac{1}{2}\left(\left(U_g \otimes U_g\right)(v_i \otimes v_j) + \left(U_g \otimes U_g\right)(v_j \otimes v_i)\right) + \frac{1}{2}\left(\left(U_g \otimes U_g\right)(v_j \otimes v_i) + \left(U_g \otimes U_g\right)(v_i \otimes v_j)\right)\right]$$
$$= \frac{1}{2}\left(\left(U_g \otimes U_g\right)(v_i \otimes v_j) + \left(U_g \otimes U_g\right)(v_j \otimes v_i)\right)$$
$$= \frac{1}{2}\left(\left(U_g(v_i) \otimes U_g(v_j)\right) + \left(U_g(v_j) \otimes U_g(v_i)\right)\right)$$
$$= \frac{1}{2}\left(\left(\lambda_i \nu_i \otimes \lambda_j \nu_j\right) + \left(\lambda_j \nu_j \otimes \lambda_i \nu_i\right)\right)$$
$$= \lambda_i \lambda_j \frac{1}{2}\left(\left(\nu_i \otimes \nu_j\right) + \left(\nu_j \otimes \nu_i\right)\right)$$

Successive equalities use: (i) linearity of $sym(U^{\otimes 2})$, (ii) definition of $sym(U^{\otimes 2})$, (iii) , combining like terms, (iv) definition of tensor product of linear transformations, (v) the fact that the v_k are eigenvectors of U_g, and (vi) linearity of the tensor product.

We could also have done this more quickly by observing that $sym(U_g^{\otimes 2}) = \left(U_g \otimes U_g\right)P$ (where P is defined above), and that $\frac{1}{2}\left(v_i \otimes v_j + v_j \otimes v_i\right)$ is an eigenvector of P with eigenvalue 1, since it is in the range of the projection.

So $\frac{1}{2}\left(v_i \otimes v_j + v_j \otimes v_i\right)$ has eigenvalue $\lambda_i \lambda_j$. We therefore have to sum up these products for all $i \le j$. Note that the case $i = j$ takes care of the eigenvectors $v_i \otimes v_i$.

$$\chi_{sym(U^{\otimes 2})}(g) = tr(sym(U_g^{\otimes 2})) = \sum_{i=1}^{n} \lambda_i^2 + \sum_{j=2}^{n} \sum_{i=1}^{j-1} \lambda_i \lambda_j .$$

This can be simplified because of the identity $\left(\sum_{i=1}^{n} \lambda_i\right)^2 = \sum_{i=1}^{n} \lambda_i^2 + 2\sum_{j=2}^{n} \sum_{i=1}^{j-1} \lambda_i \lambda_j$. Therefore,

$$\chi_{sym(U^{\otimes 2})}(g) = \frac{1}{2}\left(\sum_{i=1}^{n} \lambda_i^2 + \left(\sum_{i=1}^{n} \lambda_i\right)^2\right).$$

Finally, note that if λ_i is an eigenvalue of U_g, then λ_i^2 is an eigenvalue of $\left(U_g\right)^2$, and that $\left(U_g\right)^2 = U_{g^2}$ since U is a representation. So $\sum_{i=1}^{n} \lambda_i^2 = tr(U_{g^2}) = \chi_U(g^2)$, so

$$\chi_{sym(U^{\otimes 2})}(g) = \frac{1}{2}\left(\sum_{i=1}^{n} \lambda_i^2 + \left(\sum_{i=1}^{n} \lambda_i\right)^2\right) = \frac{1}{2}\left(\chi_U(g^2) + \left(\chi_U(g)\right)^2\right).$$

The antisymmetric part follows analogously, recognizing that the case $i = j$ is excluded. First, the eigenvalues:

$$anti(U_g^{\otimes 2})\left(\frac{1}{2}\left(v_i \otimes v_j - v_j \otimes v_i\right)\right) = \lambda_i \lambda_j \frac{1}{2}\left(\left(\nu_i \otimes \nu_j\right) - \left(\nu_j \otimes \nu_i\right)\right).$$

Then, summing these (for i strictly less than j) yields the character:

$$\chi_{anti(U^{\otimes 2})}(g) = tr(anti(U_g^{\otimes 2})) = \sum_{j=2}^{n} \sum_{i=1}^{j-1} \lambda_i \lambda_j ,$$

which similarly simplifies:

$$\chi_{anti(U^{\otimes 2})}(g) = \frac{1}{2}\left(-\sum_{i=1}^{n} \lambda_i^2 + \left(\sum_{i=1}^{n} \lambda_i\right)^2\right) = \frac{1}{2}\left(-\chi_U(g^2) + \left(\chi_U(g)\right)^2\right).$$