Linear Transformations and Group Representations

Homework #3 (2016-2017), Questions

Q1-Q2 are further exercises concerning adjoints, self-adjoint transformations, and unitary transformations. Q3-6 involve group representations. Of these, Q3 and Q4 should be quick. Q5 is especially useful for the upcoming material.

Q1: (May be skipped, and result assumed for Q2): Let L be a linear transformation, and x and y scalars.Define e^{xL} by the power series $e^{xL} = \sum_{m=0}^{\infty} \frac{1}{m!} x^m L^m$. Show that $e^{xL} e^{yL} = e^{(x+y)L}$.

Q2.A. For a general linear transformation A with adjoint A^* and (possibly complex) scalar z, show that the adjoint of zA is $\overline{z}A$.

B. Show that if A is self-adjoint and x is real (i.e, $x = \overline{x}$), then e^{ixA} is unitary, where e^{ixA} is defined as in Q1. Hint: do this by computing the adjoint of e^{ixA}.

C. An interesting special case. Recall that in a previous homework, we showed that the linear operator L, defined by $Lf(x) = i\frac{df}{dx}$ is self-adjoint in the Hilbert space of differentiable functions on the line for which $\int_{-\infty}^{\infty} |f(x)|^2 \, dx$ is finite, and the inner product ia $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$. Now we further assume that all derivatives of f exist. Apply the results of part B to $-sL$ (for s a real scalar) and develop a more familiar expression for the resulting operator.

Q3: Let $g \to U_g$ be a unitary representation of a group G .Show that $g \to \det\left(U_g\right)$ is also a unitary representation, in a vector space of dimension 1.

Q4: Let $g \to U_g$ be a unitary representation of a group G .Is $g \to \left(U_g\right)^{-1}$ a unitary representation?

[continued]

Q5. Character tables. Consider the group of rotations and mirror-flips of an equilateral triangle. Specifically, designate the three vertices as a, b, and c (in clockwise order, with a at the top), and the group operations as I for the identity, R and L for rotation right and left by 1/3 of a cycle, and M_a, M_b, and M_c for mirror flips along the lines through each of the vertices. Compute the characters at each of these elements for the representations described in the table below. With regard to S, recall (from earlier weeks) that a permutation is "odd" if it can be generated by an odd number of pair-swaps, and even if it requires an even number of pair swaps.

Group element:	I	R	L	M_a	M_b	M_c
Representation:						
E: the trivial representation (all group elements map to 1)						
P: Representation as permutation matrices on the letters $\{a,b,c\}$						
S: Representation that maps even permutations on $\{a,b,c\}$ to $+1$, odd permutations to -1						
C: Representation as 2×2 change-of-coordinate matrices in the plane						

[continued]

Q6. Character of symmetric and antisymmetric parts of a tensor product of group representations. We start with the standard setup (in the class notes) for a tensor product of group representations: U_1 a representation of G in V_1, U_2 a representation of G in V_2, leading to a group representation $U_1 \otimes U_2$ in $V_1 \otimes V_2$ whose action is defined by $\left(U_{1,g} \otimes U_{2,g}\right)(v_1 \otimes v_2) = U_{1,g}(v_1) \otimes U_{2,g}(v_2)$. Here we add to this the further supposition that $U_1 = U_2 = U$, and $V_1 = V_2 = V$. Under these circumstances, recall (see notes concerning the derivation of the determinant) that $V \otimes V$ can be decomposed into two parts: a symmetric part $sym(V^{\otimes 2})$ which has a basis consisting of elements $v_i \otimes v_i$ and $\frac{1}{2}\left(v_i \otimes v_j + v_j \otimes v_i\right)$ (for $i < j$, and v_i a basis of V), and an antisymmetric part, $anti(V^{\otimes m})$, which has a basis consisting of elements $\frac{1}{2}\left(v_i \otimes v_j - v_j \otimes v_i\right)$ (for $i < j$).

A. Show that $U_g \otimes U_g$ maps $sym(V^{\otimes 2})$ into itself and also maps $anti(V^{\otimes m})$ into itself. So $U \otimes U = U^{\otimes 2}$ can be reduced into two components, $sym(U^{\otimes 2})$ and $anti(U^{\otimes 2})$. Here it is helpful to use a coordinate-free approach, where $sym(V^{\otimes 2})$ is the range of the projection P defined by $P(v \otimes v') = \frac{1}{2}\left(v \otimes v' + v' \otimes v\right)$, and $anti(V^{\otimes 2})$ is the range of the complementary projection $I - P$.

B. Determine the characters of these two component representations ($sym(U^{\otimes 2})$ and $anti(U^{\otimes 2})$), in terms of the character of U. Here, to calculate $\chi_{sym(U^{\otimes 2})}(g) = tr(sym(U_g{}^{\otimes 2}))$ it is helpful use the bases $\frac{1}{2}\left(v_i \otimes v_j + v_j \otimes v_i\right)$ (for $i \le j$) for $sym(V^{\otimes 2})$, and to calculate $\chi_{anti(U^{\otimes 2})}(g) = tr(anti(U_g{}^{\otimes 2}))$, where the v_k are the eigenvectors of U_g. Similarly, it is helpful to use the basis $\frac{1}{2}\left(v_i \otimes v_j - v_j \otimes v_i\right)$ (for $i < j$) for $anti(V^{\otimes m})$.