Linear Transformations and Group Representations

Homework #4 (2016-2017), Answers

Q1. Dual (adjoint) representations. Recall that given a vector space V, the dual vector space V^* is the space of all linear maps from V to the base field.

A. Find a coordinate-free homomorphism between linear transformations L in $Hom(V,W)$ and linear transformations $\Omega(L)$ in $Hom(W^*,V^*)$.

For every φ in W^*, we need to find a $(\Omega(L))(\varphi)$ in V^*. That is, we need to exhibit $(\Omega(L))(\varphi)$ as a mapping from V to the base field. So we define $\left((\Omega(L))(\varphi)\right)(v)=\varphi\left(L(v)\right)$. This makes sense since $L(v)$ is in W, so φ, which is in W^*, maps $L(v)$ to a scalar.

B. Extending the above setup: Since we have several vector spaces around, let's designate the above mapping from $Hom(V,W)$ to $Hom(W^*,V^*)$ as Ω_{VW}. Correspondingly, Ω_{WX} is a mapping from $Hom(W,X)$ to $Hom(X^*,W^*)$: for every linear transformation M in $Hom(W,X)$, $\Omega_{WX}(M)$ is in $Hom(X^*,W^*)$. With this setup, ML (apply L, then apply M) is in $Hom(V,X)$, and $\Omega_{VX}(ML)$ is in $Hom(X^*,V^*)$. Show that $\Omega_{VX}(ML)=\Omega_{VW}(L)\Omega_{WX}(M)$.

For any ψ in X^* and v in V, $\left((\Omega_{VX}(ML))(\psi)\right)(v)=\psi\left(ML(v)\right)$.
But also, $\left((\Omega_{VW}(L))(\Omega_{WX}(M))(\psi)\right)(v)=\left((\Omega_{WX}(M))(\psi)\right)\left(L(v)\right)=\psi\left(ML(v)\right)$.

C. Now, taking $V=W=X$ (and $V^*=W^*=X^*$ and $\Omega=\Omega_{VW}=\Omega_{WX}=\Omega_{VX}$), and putting A and B together: we have found a mapping Ω from $Hom(V,V)$ to $Hom(V^*,V^*)$ for which $\Omega(ML)=\Omega(L)\Omega(M)$. Show that, if U_g is a representation in V, then $\Omega({U_g}^{-1})$ is a representation in V^*, and find its character. For the latter, it is useful to show that the eigenvalues of $\Omega(U)$ are the same as the eigenvalues of U, for any unitary operator U.

To show $\Omega({U_g}^{-1})$ is a representation, we need to show it preserves group structure.
$\Omega({U_{gh}}^{-1})=\Omega\left((U_gU_h)^{-1}\right)=\Omega\left({U_h}^{-1}{U_g}^{-1}\right)=\Omega\left({U_g}^{-1}\right)\Omega\left({U_h}^{-1}\right)$. The first equality is because U_g is a representation. The second is because the inverse of a product is the product of the inverses in reverse order. The third follows from part B, Ω inverts the order of multiplication.

To find its character, we first show that if A is a normal operator (i.e, it has a complete set of eigenvectors and these form a basis), then the eigenvalues of $\Omega(A)$ are the same as the eigenvalues of A. Say A has eigenvalue λ and $\Omega(A)$ has eigenvalue μ. Then $Av=\lambda v$ for some v and $\Omega(A)\varphi=\mu\varphi$ for some φ. Then $\left(\Omega(A)\varphi\right)(v)=\mu\varphi(v)$ but also

$\left(\Omega(A)\varphi\right)(v) = \varphi(Av) = \varphi(\lambda v) = \lambda\varphi(v)$. So either $\varphi(v) = 0$ or $\lambda = \mu$. For each φ, the alternative $\varphi(v) = 0$ cannot jold for all of the eigenvectors for A, since then φ would be zero on for all elements of a basis for V. So at least one eigenvector for A has $\varphi(v) \neq 0$, which in turn means that for this eigenvector, A and an eigenvector of $\Omega(A)$ share the same eigenvalue. Now strip off this eigenvector, and proceed downward.

Since the trace is the sum of the eigenvalues: $tr\left(\Omega(U_{g^{-1}})\right) = tr\left(U_{g^{-1}}\right) = tr\left(U_g^{-1}\right)$.
Since U_g is unitary, all of its eigenvalues are complex numbers of magnitude 1, i.e., complex numbers λ for which $|\lambda|^2 = \lambda\overline{\lambda} = 1$. It follows that the eigenvectors of U_g^{-1} are $\lambda^{-1} = \overline{\lambda}$. Since the trace is the sum of the eigenvectors, $tr\left(\Omega(U_{g^{-1}})\right) = \overline{tr\left(U_g\right)}$.

Q2: Find a coordinate-free homomorphism between $V^* \otimes W$ and $Hom(V,W)$. That is, for every $\varphi \otimes w$ in $V^* \otimes W$, find an element $\Phi = Z(\varphi \otimes w)$ in $Hom(V,W)$, such that the mapping Z from $\varphi \otimes w$ to Φ is linear. (See Q2 of Homework #3, Groups, Fields and Vector Spaces (2008-2009) for more of this type.)

To exhibit $\Phi = Z(\varphi \otimes w)$ as a member of $Hom(V,W)$, we need to demonstrate it as a linear transformation from elements v of V into elements w of W. We are given φ in V^*, so it is a linear map from V to the base field k. Therefore, $\varphi(v)$ is a scalar, and $\varphi(v)w$ is in W. So we can define $Z(\varphi \otimes w)$ as the homomorphism Φ for which $\Phi(v) = \varphi(v)w$.

Q3. Character tables. Consider the group of rotations and mirror-flips of a square. Specifically, designate the three vertices as a, b, c, and d (in clockwise order, with a at the top right), and the group operations as I for the identity; R and L for rotation right and left by 1/4 of a cycle; Z for rotation by 1/2 of a cycle, M_V for a mirror flip on the vertical axis (swapping $a \leftrightarrow d$ and $b \leftrightarrow c$); M_H for a mirror flip on the vertical axis (swapping $a \leftrightarrow b$ and $c \leftrightarrow d$), M_{ac} a flip on the diagonal running from a to c (swapping $b \leftrightarrow d$), and M_{bd} a flip on the diagonal running from b to d (swapping $a \leftrightarrow c$). Compute the characters at each of these elements for the representations described in the table below. Recall (from earlier weeks) that a permutation is "odd" if it can be generated by an odd number of pair-swaps, and even if it requires an even number of pair swaps.

Group element:	I	R	L	Z	M_V	M_H	M_{ac}	M_{bd}
Representation:								
E : the trivial representation (all group elements map to 1)								
P : Representation as permutation matrices								

on the letters $\{a,b,c,d\}$

S *: Representation that maps even permutations on* $\{a,b,c,d\}$

P_{opp} *: Representation as permutation matrices on the two pairs of opposite sides*

P_{diag} *: Representation as permutation matrices on the two diagonals*

C *: Representation as* 2×2 *change-of-coordinate matrices in the plane*

R *:Regular representation*

The completed table follows this analysis:

Group element:	I	R	L	Z	M_V	M_H	M_{ac}	M_{bd}
E *: the trivial representation*	1	1	1	1	1	1	1	1
P *: permutations on* $\{a,b,c,d\}$	4	0	0	0	0	0	2	2
S *: even and odd permutations on* $\{a,b,c,d\}$	1	-1	-1	1	1	1	-1	-1
P_{diag} *: permutation matrices on the two diagonals*	2	0	0	2	0	0	2	2
P_{opp} *: permutation matrices on the two pairs of opposite sides*	2	0	0	2	2	2	0	0
C *: Representation as* 2×2 *change-of-coordinate matrices in the plane*	2	0	0	-2	0	0	0	0
R *: Regular representation*	8	0	0	0	0	0	0	0