Linear Transformations and Group Representations

Homework #4 (2016-2017), Questions

Q1. Dual (adjoint) representations. Recall that given a vector space V, the dual vector space V^* is the space of all linear maps from V to the base field.

A. Find a coordinate-free homomorphism between linear transformations L in $Hom(V,W)$ and linear transformations $\Omega(L)$ in $Hom(W^*,V^*)$.

B. Extending the above setup: Since we have several vector spaces around, let's designate the above mapping from $Hom(V,W)$ to $Hom(W^*,V^*)$ as Ω_{VW}. Correspondingly, Ω_{WX} is a mapping from $Hom(W,X)$ to $Hom(X^*,W^*)$: for every linear transformation M in $Hom(W,X)$, $\Omega_{WX}(M)$ is in $Hom(X^*,W^*)$. With this setup, ML (apply L, then apply M) is in $Hom(V,X)$, and $\Omega_{VX}(ML)$ is in $Hom(X^*,V^*)$. Show that $\Omega_{VX}(ML)=\Omega_{VW}(L)\Omega_{WX}(M)$.

C. Now, taking $V=W=X$ (and $V^*=W^*=X^*$), and putting A and B together: we have found a mapping Ω from $Hom(V,V)$ to $Hom(V^*,V^*)$ for which $\Omega(ML)=\Omega(L)\Omega(M)$. Show that, if U_g is a representation in V, then $\Omega(U_g{}^{-1})$ is a representation in V^*, and find its character. For the latter, it is useful to show that the eigenvalues of $\Omega(U)$ are the same as the eigenvalues of U, for any unitary operator U.

Q2: Find a coordinate-free homomorphism between $V^*\otimes W$ and $Hom(V,W)$. That is, for every $\varphi\otimes w$ in $V^*\otimes W$, find an element $\Phi=Z(\varphi\otimes w)$ in $Hom(V,W)$, such that the mapping Z from $\varphi\otimes w$ to Φ is linear. (See Q2 of Homework #3, Groups, Fields and Vector Spaces (2008-2009) for more of this type.)

Q3. Character tables. Consider the group of rotations and mirror-flips of a square. Specifically, designate the three vertices as a, b, c, and d (in clockwise order, with a at the top right), and the group operations as I for the identity; R and L for rotation right and left by 1/4 of a cycle; Z for rotation by 1/2 of a cycle, M_V for a mirror flip on the vertical axis (swapping $a\leftrightarrow d$ and $b\leftrightarrow c$); M_H for a mirror flip on the vertical axis (swapping $a\leftrightarrow b$ and $c\leftrightarrow d$), M_{ac} a flip on the diagonal running from a to c (swapping $b\leftrightarrow d$), and M_{bd} a flip on the diagonal running from b to d (swapping $a\leftrightarrow c$). Compute the characters at each of these elements for the representations described in the table below. Recall (from earlier weeks) that a permutation is "odd" if it can be generated by an odd number of pair-swaps, and even if it requires an even number of pair swaps.

Group element:	I	R	L	Z	M_V	M_H	M_{ac}	M_{bd}
Representation:								

E : the trivial representation
(all group elements map to 1)

P : Representation as
permutation matrices
on the letters $\{a,b,c,d\}$

S : Representation that maps
even permutations on $\{a,b,c,d\}$

P_{opp} : Representation as
permutation matrices on
the two pairs of opposite sides

P_{diag} : Representation as
permutation matrices
on the two diagonals

C : Representation as
2×2 change-of-coordinate
matrices in the plane

R :Regular representation