Exam, 2016-2017 Solutions

Note that many of the answers are more detailed than is required for "full credit"

Question 1. The Frobenius norm

The Frobenius norm of a matrix M, $\|M\|_F$ which is an overall measure of the size of the matrix, is defined as the square root of the sum of the squares of the absolute values of its entries, i.e., $\|M\|_F = \sqrt{\sum_{i,j} |m_{i,j}|^2}$.

A. Show that $\|M\|_F = \sqrt{tr\left(MM^*\right)}$, where M^* is the adjoint of M .

For any matrix M with elements $\left(M\right)_{i,j} = m_{ij}$, the elements of the adjoint are $\left(M^*\right)_{i,j} = \overline{m_{ji}}$. So the element in position (j,k) of the product MM^* are $\left(MM^*\right)_{i,k} = \sum_j \left(M\right)_{i,j}\left(M^*\right)_{j,k} = \sum_j m_{ij}\overline{m_{kj}}$. So the trace of MM^*, which is the sum of the elements on the diagonal, is given by $tr\left(MM^*\right) = \sum_i \left(MM^*\right)_{i,i} = \sum_i \left(\sum_j m_{ij}\overline{m_{ij}}\right) = \sum_{i,j} |m_{i,j}|^2$ .Therefore, $\|M\|_F = \sqrt{\sum_{i,j} |m_{i,j}|^2} = \sqrt{tr\left(MM^*\right)}$

B. If A is self-adjoint, show that $\|A\|_F = \sqrt{\sum \lambda_i^{\ 2}}$, where the λ_i are the eigenvalues of A.

If A is self-adjoint, then $\|A\|_F = \sqrt{tr\left(AA^*\right)} = \sqrt{tr\left(A^2\right)}$. The trace is the sum of the eigenvalues, and the eigenvalues of the square of a matrix are the squares of its eigenvalues (since if $Av = \lambda v$, then $A^2 v = A(Av) = A(\lambda v) = \lambda Av = \lambda^2 v$).

C. Show that if U is unitary, then $\|M\|_F = \|UMU^{-1}\|_F$, i.e., the Frobenius norm is invariant under unitary transformation.

$\|UMU^{-1}\|_F = \sqrt{tr\left[\left(UMU^{-1}\right)\left(UMU^{-1}\right)^*\right]}$. We work out the adjoint of UMU^{-1} by taking the adjoint of the terms, in reverse order: $\left(UMU^{-1}\right)^* = \left(U^{-1}\right)^* M^* U^*$. Since U is unitary, $U^{-1} = U^*$. So $\left(UMU^{-1}\right)^* = \left(U^{-1}\right)^* M^* U^* = UM^*U^{-1}$, and

$$\|UMU^{-1}\|_F = \sqrt{tr\left[\left(UMU^{-1}\right)\left(UMU^{-1}\right)^*\right]} = \sqrt{tr\left[\left(UMU^{-1}\right)\left(UM^*U^{-1}\right)\right]}$$
$$= \sqrt{tr\left(UMU^{-1}UM^*U^{-1}\right)} = \sqrt{tr\left(UMM^*U^{-1}\right)}.$$

Finally, since the trace is independent of coordinates (or, since $tr(XY) = tr(YX)$ for any two matrices X and Y),
$\left\|UMU^{-1}\right\|_F = \sqrt{tr\left(UMM^*U^{-1}\right)} = \sqrt{tr\left(MM^*\right)} = \left\|M\right\|_F$.

Question 2. Representations one group inside of another group

Say G is a (finite) group, U is a representation (with $U : g \to U_g$ a homomorphism from G to unitary transformations in $Hom(V,V)$, and H is a proper subgroup G (i.e., H is a subgroup and $H \neq G$).

A. Show that U is a representation of H.
Since $U : g \to U_g$ is a mapping from G into unitary transformations for which $U_{gg'} = U_g U_{g'}$, it necessarily has these properties for a subset of G.

B. Show that if U is an irreducible representation of H, then it is also an irreducible representation of G.

If U is irreducible on H, then (according to the definition of irreducibility) there is no direct-sum decomposition $V = V_1 \oplus V_2$ in which all of the transformations U_h act separately in V_1 and V_2 (i.e., there is no basis in which all of the matrices U_h have the same block-diagonal form), for all $h \in H$. Since G includes all of H, these conditions necessarily hold for U considered as a representation of G.

C. If U is an irreducible representation of G, then is it necessarily an irreducible representation of H? If yes, provide a proof; if no, provide an example in which U becomes reducible when restricted to a subgroup.
Typically, even if U *is irreducible on* G, *it is reducible on* H. As an example, take $G = S_3$, the permutation group on three objects S_3, or, equivalently, the rotations and reflections of an equilateral triangle. It has an irreducible representation of dimension 2, corresponding to these matrix transformations. It also has a subgroup H consisting of just the rotations. This is a commutative group, so all of its representations are necessarily one-dimensional.

D. Given the above setup, with U an irreducible representation of G, and further assume that the character of U is zero for all elements of G that are not in H (this behavior is not unusual, for example see part E of Q3 below). Then show that U is always reducible on H.
If U_H (i.e., U restricted to H), is irreducible, then $d(U_H, U_H) = 1$. By the trace formula,

$$d(U_H,U_H)=\frac{1}{|H|}\sum_{g\in H}|\chi_U(g)|^2$$
$$=\frac{|G|}{|H|}\left(\frac{1}{|G|}\sum_{g\in H}|\chi_U(g)|^2\right)=\frac{|G|}{|H|}\left(\frac{1}{|G|}\left[\sum_{g\in H}|\chi_U(g)|^2+\sum_{g\notin H}|\chi_U(g)|^2\right]\right),$$
$$=\frac{|G|}{|H|}\left(\frac{1}{|G|}\sum_{g\in G}|\chi_U(g)|^2\right)=\frac{|G|}{|H|}d(U,U)=\frac{|G|}{|H|}$$
The last two equalities follow from the trace formula and the hypothesis that U an irreducible representation of G. Since $|H|<|G|$ (because H is a proper subgroup G), the last quantity is larger than 1.

Question 3. Another way to construct group representations

Here we will develop another way to construct group representations. Let G be a (finite) group, H is a subgroup, and recall that, for any $b\in G$ a "right coset" of H, denoted Hb, is the set of all elements of $g\in G$ that can be written as $g=hb$. We showed in class that G is the disjoint union of all of its distinct right cosets (including the right coset corresponding to the identity $H=He$). This means that the number of right cosets is $|G|/|H|$. Any element $c\in G$ can be viewed as acting as a permutation on the cosets, since $(Hb)c$ is also a coset, namely, $(Hb)c=H(bc)$.

A. Show that the action on cosets yields a permutation representation of G, that we will denote here as Q.

We need to show that the mapping from group elements to permutations of cosets is a homomorphism from the group G to the permutations on cosets; if so, then $Q:g\to Q_g$ will be the mapping from G into the corresponding permutation matrices. To show homomorphism, we need to show that the permutation corresponding to the group element c_1, followed by the permutation corresponding to the group element c_2, is the same as the permutation corresponding to the group element c_1c_2. This follows from associativity:
$((Hb)c_1)c_2=(Hbc_1)c_2=H(bc_1c_2)=Hb(c_1c_2)$.

B. What does the above construction reduce to if $H=G$? What does it reduce to if H is just the identity element?

If $H=G$, then there is only one coset, H itself, which is the entire group. So the construction yields the trivial one-dimensional representation that maps every group element to 1.

If $H=\{e\}$, then every element of G is its own coset, so the construction yields the regular representation.

C. Assume that H is is not all of G, i.e., that there is at least some element of G that is not in H. Show that the representation we constructed is NOT irreducible.

First, note that since every Q_g is a permutation matrix, its trace $\chi_Q(g)$ must be a non-negative integer, and that $\chi_Q(e) = \frac{|G|}{|H|}$, the number of cosets. We apply the trace formula, to show that Q contains at least one copy of the trivial representation I, for which $I_g = 1$:

$d(Q,I) = \frac{1}{|G|}\sum_{g\in G}\chi_Q(g)\overline{\chi_I(g)} = \frac{1}{|G|}\sum_{g\in G}\chi_Q(g)$ (since for the trivial representation, $\chi_I(g) = 1$). This expression must be an integer. Since $\chi_Q(e) > 0$ and all of the other character terms cannot be negative, the final right hand side cannot be zero. So $d(Q,I) \geq 1$, which means that that Q contains at least one copy of the trivial representation I,

It remains to be shown that Q maps at least one group element to something that is not the identity. We show this by choosing some $g \notin H$. If it were the case that Q_g is the identity permutation, then Q_g would map the coset H to itself, which means that $Hg = H$. This means that $hg \in H$ for every h. , which implies $hg \in H \Rightarrow hg = h' \Rightarrow g = h^{-1}h' \Rightarrow g \in H$, a contradiction.

D. Now assume that H is a normal subgroup of G. (A "normal" subgroup is a subgroup for which every right coset is also a left coset, i.e., that $bH = Hb$. Equivalently, for a normal subgroup H and any $h \in H$ and any $g \in G$, then $ghg^{-1} \in H$.) Determine the character of $\chi_Q(h)$ for $h \in H$.

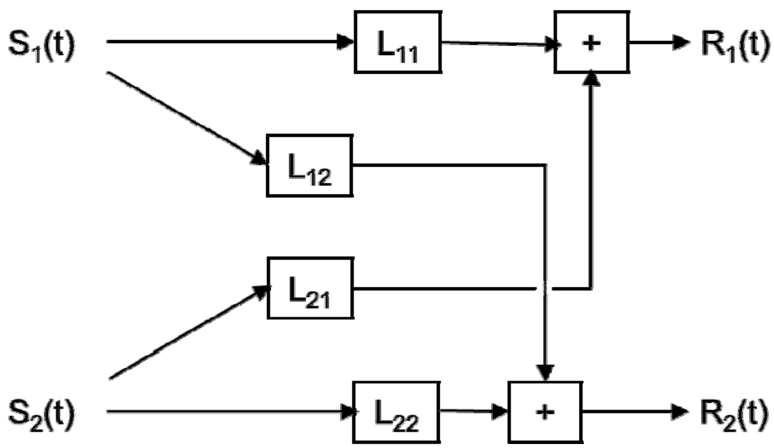
We show that the permutation corresponding to right multiplication of cosets by h leaves all cosets unchanged. That is, for any coset Hb, $Hbh = Hb$. To do this, note that for any $h_1bh \in Hbh$, $h_1bh = h_1bhb^{-1}b = h_1(bhb^{-1})b$. The middle term bhb^{-1} is in H because H is assumed normal. So the right hand side is a member of Hb. Since right multiplication of cosets by h leaves all cosets unchanged, Q_h is the identity permutation. Its dimension is the number of cosets, which is $|G|/|H|$. So its trace, which is the character $\chi_Q(h)$, is $|G|/|H|$.

E. As in D, but now determine the character $\chi_Q(c)$ for $c \notin H$.

We show that the permutation corresponding to right multiplication of cosets by $c \notin H$ moves every coset to a different one. That is we show that for any coset Hb, $Hbc \neq Hb$. To do this, assume the contrary and note that if any $h_1bc \in Hbc$ and any $h_2b \in Hb$ were equal, then $h_1bc = h_2b \Rightarrow bc = h_1^{-1}h_2b \Rightarrow c = b^{-1}h_1^{-1}h_2b$. The last quantity is in H because H is assumed normal. So this is a contradiction, and hence $Hbc \neq Hb$.. Since right multiplication of cosets by c moves every coset to a distinct one, Q_c is a permutation with no 1's on its diagonal. So its trace, which is the character $\chi_Q(c)$, is 0.

Question 4. Coherence and network identification

Say $S_1(t)$ and $S_2(t)$ are independent noise sources with power spectra $P_{S_1}(\omega)$ and $P_{S_2}(\omega)$, which are connected to two observable outputs $R_1(t)$ and $R_2(t)$ by the following network,where L_{ij} are linear filters with transfer functions $\tilde{L}_{ij}(\omega)$.



A. Find the power spectra $P_{R_1}(\omega)$ and $P_{R_2}(\omega)$ of the two outputs.

Let $\tilde{s}_i(\omega)$ be Fourier transforms of the input signals over some time interval T, and let $\tilde{r}_k(\omega)$ be Fourier transforms of the output signals over this interval. Because of the definition of the power spectrum, $\lim_{T\to\infty}\frac{1}{T}\left\langle \tilde{s}_i(\omega)\overline{\tilde{s}_i(\omega)}\right\rangle = P_{S_i}(\omega)$. Because the two inputs are independent,

$\lim_{T\to\infty}\frac{1}{T}\left\langle \tilde{s}_i(\omega)\overline{\tilde{s}_j(\omega)}\right\rangle = 0$ for $i \neq j$. Because of the network, $\tilde{r}_k(\omega) = \sum_i \tilde{L}_{ik}(\omega)\tilde{s}_i(\omega)$. Therefore,

$$P_{R_k}(\omega) = \lim_{T\to\infty}\frac{1}{T}\left\langle \tilde{r}_k(\omega)\overline{\tilde{r}_k(\omega)}\right\rangle = \lim_{T\to\infty}\frac{1}{T}\left\langle \left(\sum_i \tilde{L}_{ik}(\omega)\tilde{s}_i(\omega)\right)\left(\overline{\sum_j \tilde{L}_{jk}(\omega)\tilde{s}_j(\omega)}\right)\right\rangle$$

$$= \lim_{T\to\infty}\frac{1}{T}\left\langle \sum_{i,j} \tilde{L}_{ik}(\omega)\tilde{s}_i(\omega)\overline{\tilde{L}_{jk}(\omega)\tilde{s}_j(\omega)}\right\rangle = \lim_{T\to\infty}\frac{1}{T}\left\langle \sum_{i,j} \tilde{L}_{ik}(\omega)\overline{\tilde{L}_{jk}(\omega)}\tilde{s}_i(\omega)\overline{\tilde{s}_j(\omega)}\right\rangle .$$

$$= \lim_{T\to\infty}\frac{1}{T}\left\langle \sum_i \tilde{L}_{ik}(\omega)\overline{\tilde{L}_{ik}(\omega)}\tilde{s}_i(\omega)\overline{\tilde{s}_i(\omega)}\right\rangle = \sum_i \tilde{L}_{ik}(\omega)\overline{\tilde{L}_{ik}(\omega)}\left\langle \lim_{T\to\infty}\frac{1}{T}\tilde{s}_i(\omega)\overline{\tilde{s}_i(\omega)}\right\rangle$$

$$= \sum_i \left|\tilde{L}_{ik}(\omega)\right|^2 P_{S_i}(\omega)$$

B. Find the cross-spectrum of $R_1(t)$ and $R_2(t)$.

$$P_{R_1R_2}(\omega) = \lim_{T\to\infty} \frac{1}{T}\left\langle \tilde{r}_1(\omega)\overline{\tilde{r}_2(\omega)} \right\rangle = \lim_{T\to\infty} \frac{1}{T}\left\langle \left(\sum_i \tilde{L}_{i1}(\omega)\tilde{s}_i(\omega)\right)\left(\overline{\sum_j \tilde{L}_{j2}(\omega)\tilde{s}_j(\omega)}\right) \right\rangle$$

$$= \lim_{T\to\infty} \frac{1}{T}\left\langle \sum_{i,j} \tilde{L}_{i1}(\omega)\tilde{s}_i(\omega)\overline{\tilde{L}_{j2}(\omega)\tilde{s}_j(\omega)} \right\rangle = \lim_{T\to\infty} \frac{1}{T}\left\langle \sum_{i,j} \tilde{L}_{i1}(\omega)\overline{\tilde{L}_{j2}(\omega)}\tilde{s}_i(\omega)\overline{\tilde{s}_j(\omega)} \right\rangle$$

$$= \lim_{T\to\infty} \frac{1}{T}\left\langle \sum_{i} \tilde{L}_{i1}(\omega)\overline{\tilde{L}_{i2}(\omega)}\tilde{s}_i(\omega)\overline{\tilde{s}_i(\omega)} \right\rangle = \sum_i \tilde{L}_{i1}(\omega)\overline{\tilde{L}_{i2}(\omega)}\left\langle \lim_{T\to\infty} \frac{1}{T}\tilde{s}_i(\omega)\overline{\tilde{s}_i(\omega)} \right\rangle$$

$$= \sum_i \tilde{L}_{i1}(\omega)\overline{\tilde{L}_{i2}(\omega)}P_{S_i}(\omega) = \tilde{L}_{11}(\omega)\overline{\tilde{L}_{12}(\omega)}P_{S_1}(\omega) + \tilde{L}_{21}(\omega)\overline{\tilde{L}_{22}(\omega)}P_{S_2}(\omega)$$

C. Find the coherence of $R_1(t)$ *and* $R_2(t)$.

$$C_{R_1R_2}(\omega) = \frac{P_{R_1R_2}(\omega)}{\sqrt{P_{R_1}(\omega)P_{R_2}(\omega)}} = \frac{\tilde{L}_{11}(\omega)\overline{\tilde{L}_{12}(\omega)}P_{S_1}(\omega) + \tilde{L}_{21}(\omega)\overline{\tilde{L}_{22}(\omega)}P_{S_2}(\omega)}{\sqrt{\left(\left|\tilde{L}_{11}(\omega)\right|^2 P_{S_1}(\omega) + \left|\tilde{L}_{21}(\omega)\right|^2 P_{S_2}(\omega)\right)\left(\left|\tilde{L}_{12}(\omega)\right|^2 P_{S_1}(\omega) + \left|\tilde{L}_{22}(\omega)\right|^2 P_{S_2}(\omega)\right)}}$$

D. When is the magnitude of the coherence equal to 1?

The magnitude-squared of the numerator of the coherence (part D), obtained by multiplying the numerator by its complex conjugate, is

$$\left(\tilde{L}_{11}(\omega)\overline{\tilde{L}_{12}(\omega)}P_{S_1}(\omega) + \tilde{L}_{21}(\omega)\overline{\tilde{L}_{22}(\omega)}P_{S_2}(\omega)\right)\left(\overline{\tilde{L}_{11}(\omega)}\tilde{L}_{12}(\omega)P_{S_1}(\omega) + \overline{\tilde{L}_{21}(\omega)}\tilde{L}_{22}(\omega)P_{S_2}(\omega)\right)$$

$$= \left|\tilde{L}_{11}(\omega)\right|^2\left|\tilde{L}_{12}(\omega)\right|^2 P_{S_1}(\omega)^2 + \left|\tilde{L}_{21}(\omega)\right|^2\left|\tilde{L}_{22}(\omega)\right|^2 P_{S_2}(\omega)^2 +$$

$$P_{S_1}(\omega)P_{S_2}(\omega)\left(\tilde{L}_{11}(\omega)\overline{\tilde{L}_{12}(\omega)}\overline{\tilde{L}_{21}(\omega)}\tilde{L}_{22}(\omega) + \tilde{L}_{21}(\omega)\overline{\tilde{L}_{22}(\omega)}\overline{\tilde{L}_{11}(\omega)}\tilde{L}_{12}(\omega)\right).$$

If the coherence is1, this must equal the magnitude-squared of the denominator. The difference between the above quantity and the magnitude-squared of the denominator is

$$\Delta = P_{S_1}(\omega)P_{S_2}(\omega)\left(\tilde{L}_{11}(\omega)\overline{\tilde{L}_{12}(\omega)}\overline{\tilde{L}_{21}(\omega)}\tilde{L}_{22}(\omega) + \tilde{L}_{21}(\omega)\overline{\tilde{L}_{22}(\omega)}\overline{\tilde{L}_{11}(\omega)}\tilde{L}_{12}(\omega)\right) -$$

$$P_{S_1}(\omega)P_{S_2}(\omega)\left(\left|\tilde{L}_{11}(\omega)\right|^2\left|\tilde{L}_{22}(\omega)\right|^2 + \left|\tilde{L}_{21}(\omega)\right|^2\left|\tilde{L}_{12}(\omega)\right|^2\right)$$

$$= P_{S_1}(\omega)P_{S_2}(\omega)\left(\tilde{L}_{11}(\omega)\overline{\tilde{L}_{12}(\omega)}\overline{\tilde{L}_{21}(\omega)}\tilde{L}_{22}(\omega) + \tilde{L}_{21}(\omega)\overline{\tilde{L}_{22}(\omega)}\overline{\tilde{L}_{11}(\omega)}\tilde{L}_{12}(\omega)\right.$$

$$\left. - \tilde{L}_{11}(\omega)\overline{\tilde{L}_{11}(\omega)}\overline{\tilde{L}_{22}(\omega)}\tilde{L}_{22}(\omega) - \tilde{L}_{12}(\omega)\overline{\tilde{L}_{12}(\omega)}\overline{\tilde{L}_{21}(\omega)}\tilde{L}_{21}(\omega)\right).$$

$$= P_{S_1}(\omega)P_{S_2}(\omega)\left(\tilde{L}_{11}(\omega)\tilde{L}_{22}(\omega) - \tilde{L}_{12}(\omega)\tilde{L}_{21}(\omega)\right)\left(\overline{-\tilde{L}_{11}(\omega)\tilde{L}_{22}(\omega) + \tilde{L}_{12}(\omega)\tilde{L}_{21}(\omega)}\right)$$

$$= -P_{S_1}(\omega)P_{S_2}(\omega)\left|\tilde{L}_{11}(\omega)\tilde{L}_{22}(\omega) - \tilde{L}_{12}(\omega)\tilde{L}_{21}(\omega)\right|^2$$

The coherence is 1 when the above quantity is zero.

E. Provide an interpretation for the answer in D.

For the cross-spectrum to be zero, at least one of the three terms in the above expression must be zero. That is, either there is only one input, or, $\tilde{L}_{11}(\omega)\tilde{L}_{22}(\omega) - \tilde{L}_{12}(\omega)\tilde{L}_{21}(\omega) = 0$. In the latter case,

$\frac{\tilde{L}_{21}(\omega)}{\tilde{L}_{11}(\omega)} = \frac{\tilde{L}_{22}(\omega)}{\tilde{L}_{12}(\omega)}$ Say this common quotient is $\tilde{q}(\omega)$. It follows that

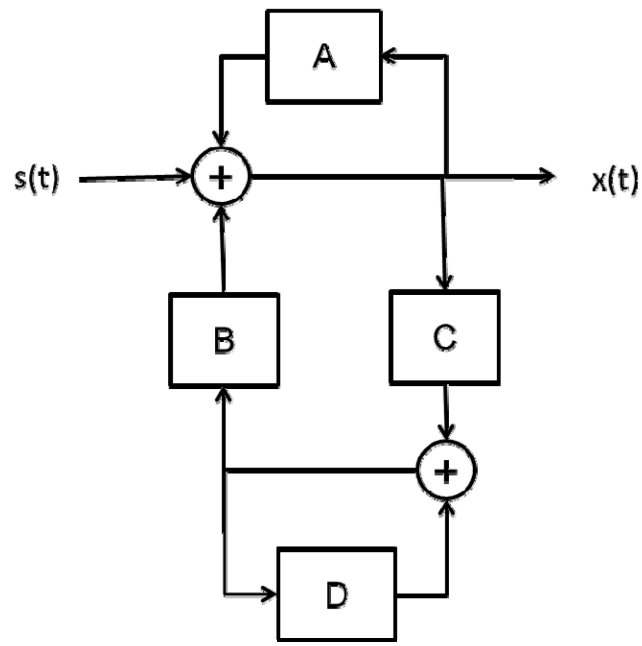
$$\tilde{r}_k(\omega) = \sum_i \tilde{L}_{ik}(\omega)\tilde{s}_i(\omega) = \tilde{L}_{1k}(\omega)\tilde{s}_1(\omega) + \tilde{L}_{1k}(\omega)\tilde{q}(\omega)\tilde{s}_1(\omega),$$
$$= \tilde{L}_{1k}(\omega)(1+\tilde{q}(\omega))\tilde{s}_1(\omega)$$

i.e., that the system is indistinguishable from one in which $R_1(t)$ and $R_2(t)$ can be viewed as being derived from a single common signal.

Question 5. Coupled neurons (or populations): linear systems view

Consider the network below, in which one neural population (the "+" at the top) is self-exciting with linear dynamics specified by the impulse response $A(t)$*, a second neural population (the "+" at the bottom) is self-exciting with linear dynamics specified by* $D(t)$*, and the two neurons are coupled to each other via the linear dynamics specified by the impulse responses* $C(t)$ *and* $B(t)$*.*

Determine the transfer function between the input $s(t)$ *and the output* $x(t)$*, in terms of the transfer functions* $\tilde{A}(\omega)$*,* $\tilde{B}(\omega)$*,* $\tilde{C}(\omega)$*, and* $\tilde{D}(\omega)$*.*



Solution:

Looking at the output of A: $\tilde{a}(\omega) = \tilde{A}(\omega)\tilde{x}(\omega)$. Looking at the output of C:. $\tilde{c}(\omega) = \tilde{C}(\omega)\tilde{x}(\omega)$.

Looking at the output of D:

$$\tilde{d}(\omega) = \tilde{D}(\omega)\tilde{y}(\omega)$$
$$= \tilde{D}(\omega)\left(\tilde{c}(\omega) + \tilde{d}(\omega)\right)$$
$$= \tilde{D}(\omega)\left(\tilde{C}(\omega)\tilde{x}(\omega) + \tilde{d}(\omega)\right)$$

which implies

$$\tilde{d}(\omega) = \frac{\tilde{C}(\omega)\tilde{D}(\omega)}{1-\tilde{D}(\omega)}\tilde{x}(\omega).$$

Looking at the output of B:

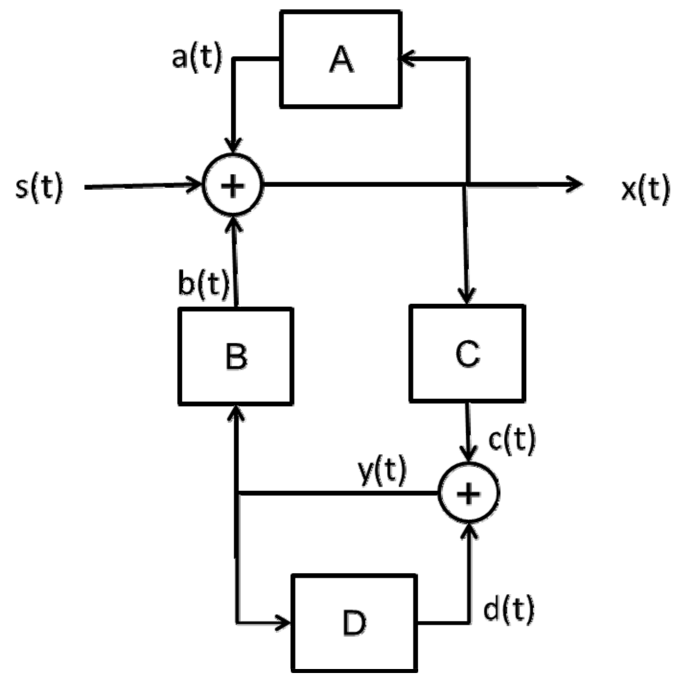
$$\tilde{b}(\omega) = \tilde{B}(\omega)\tilde{y}(\omega)$$

$$= \tilde{B}(\omega)\left(\tilde{c}(\omega) + \tilde{d}(\omega)\right)$$

$$= \tilde{B}(\omega)\left(\tilde{C}(\omega)\tilde{x}(\omega) + \frac{\tilde{C}(\omega)\tilde{D}(\omega)}{1-\tilde{D}(\omega)}\tilde{x}(\omega)\right)$$

$$= \tilde{B}(\omega)\left(\tilde{C}(\omega) + \frac{\tilde{C}(\omega)\tilde{D}(\omega)}{1-\tilde{D}(\omega)}\right)\tilde{x}(\omega)$$

$$= \frac{\tilde{B}(\omega)\tilde{C}(\omega)}{1-\tilde{D}(\omega)}\tilde{x}(\omega)$$



Looking at the three signals that sum to form $x(t)$:

$$\tilde{x}(\omega) = \tilde{b}(\omega) + \tilde{s}(\omega) + \tilde{A}(\omega)\tilde{x}(\omega)$$

$$= \frac{\tilde{B}(\omega)\tilde{C}(\omega)}{1-\tilde{D}(\omega)}\tilde{x}(\omega) + \tilde{s}(\omega) + \tilde{A}(\omega)\tilde{x}(\omega),$$

which implies that

$$\tilde{x}(\omega)\left(1 - \frac{\tilde{B}(\omega)\tilde{C}(\omega)}{1-\tilde{D}(\omega)} - \tilde{A}(\omega)\right) = \tilde{s}(\omega).$$

So the transfer function that relates $x(t)$ to $s(t)$ is:

$$\frac{\tilde{x}(\omega)}{\tilde{s}(\omega)} = \frac{1}{1 - \frac{\tilde{B}(\omega)\tilde{C}(\omega)}{1-\tilde{D}(\omega)} - \tilde{A}(\omega)}.$$

Question 6. Coupled neurons (or neural populations), dynamical systems view

Let $x(t)$ and $y(t)$ represent the fluctuations in activity of two neural populations; for convenience, we set the mean level of both to zero. Consider the following dynamics:

$$\frac{dx}{dt} = ax - by - Ax^3$$
$$\frac{dy}{dt} = -cx + dy$$

, where $a > 0$, $d > 0$ (the populations are self-exciting), $b > 0$, $c > 0$ (so the populations inhibit each other), and $A > 0$ (so there is no runaway activity in either direction of $x(t)$).

A. What are the possible kinds of behavior near $(x, y) = (0, 0)$, and for which parameter values do they occur?

Solution:

$(0,0)$ is a fixed point, and the Jacobian of the linearized system is $J=\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$. Is eigenvalues are roots of the its equation, $\lambda^2-\lambda\,\text{trace}(J)+\det(J)=0$, i.e.,

$\lambda^2-\lambda(a+d)+(ad-bc)=0$. These roots are given by

$$\lambda=\frac{(a+d)\pm\sqrt{(a+d)^2-4(ad-bc)}}{2}$$

$$=\frac{(a+d)\pm\sqrt{(a-d)^2+4bc}}{2}.$$

Since b and c are both positive, the expression under the radical must be real. So both eigenvalues of J are real. Since they sum to $a+d$, a positive number, they must either be both positive, or one positive and one negative. The transition occurs when the smaller root is zero, namely,

$$\frac{(a+d)\pm\sqrt{(a-d)^2+4bc}}{2}=0$$

$\Rightarrow(a+d)^2=(a-d)^2+4bc$.(This could also have been seen directly, since a zero eigenvalue implies

$\Rightarrow ad=bc$

that $\det(J)=ad-bc=0$).

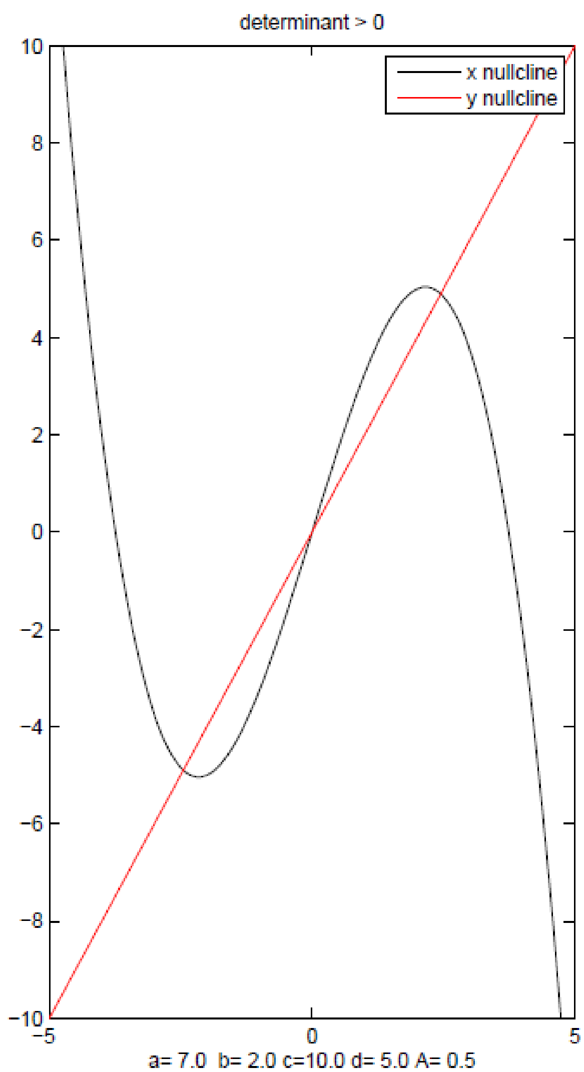
So one kind of behavior is that $(0,0)$ is an unstable node; another is that it is a saddle point, with the first occurring if $ad>bc$ -- where the self-excitation terms dominate the mutual inhibition terms.

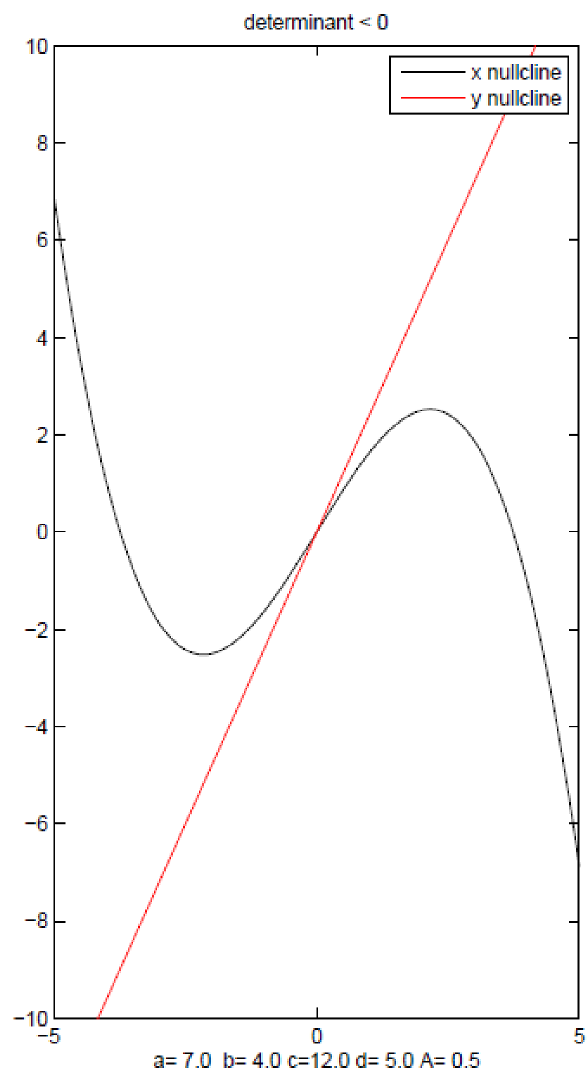
B. Sketch the nullclines (the loci in the (x,y) plane in which $\frac{dx}{dt}=0$ and $\frac{dy}{dt}=0$) for the regimes identified in A.

The nullcline for $\frac{dx}{dt}=0$ is $0=ax-by-Ax^3$, i.e., $y=\frac{a}{b}x-\frac{A}{b}x^3$. This is a cubic that ascends near the origin with (positive) slope $\frac{a}{b}$, and then eventually descends for large $x>0$. The nullcline for $\frac{dy}{dt}=0$ is $0=cx-dy$, i.e., $y=\frac{c}{d}x$, a line through the origin with (positive) slope $\frac{c}{d}$. If the determinant $\det J=ad-bc$ is positive, the slope of the cubic is larger: $ad-bc>0$ which implies $\frac{a}{b}>\frac{c}{d}$. This is shown in the left panel below. Conversely, if $\det J<0$, the slope of the line is larger, as $\frac{a}{b}<\frac{c}{d}$. This is in the right panel below.

```
figure;set(gcf,'Position',[50 50 1000 700]);
xv=[-5:.1:5];
%
subplot(1,2,1);
a=7;b=2;c=10;d=5;A=0.5;
nullx=(1/b)*[a*xv-A*xv.^3];
nully=(1/d)*c*xv;
plot(xv,nullx,'k');hold on;
plot(xv,nully,'r');hold on;
title('determinant > 0');
```

```
legend('x nullcline','y nullcline');
set(gca,'XLim',[-5 5]);
set(gca,'YLim',[-10 10]);
xlabel(sprintf('a=%4.1f  b=%4.1f c=%4.1f d=%4.1f A=%4.1f',a,b,c,d,A'));
%
subplot(1,2,2);
a=7;b=4;c=12;d=5;A=0.5;
nullx=(1/b)*[a*xv-A*xv.^3];
nully=(1/d)*c*xv;
plot(xv,nullx,'k');hold on;
plot(xv,nully,'r');hold on;
title('determinant < 0');
legend('x nullcline','y nullcline');
set(gca,'XLim',[-5 5]);
set(gca,'YLim',[-10 10]);
xlabel(sprintf('a=%4.1f  b=%4.1f c=%4.1f d=%4.1f A=%4.1f',a,b,c,d,A'));
```





C. Are there any other fixed points for the regimes identified in A? Under what circumstances?

The full set of fixed points are the solutions of $\frac{dx}{dt} = \frac{dy}{dt} = 0$, i.e., solutions of the two equations

$$\begin{aligned} 0 &= ax - by - Ax^3 \\ 0 &= -cx + dy \end{aligned}$$

. From the second equation, $y = cx/d$. Substituting into the first equation:

$0 = ax - \frac{bcx}{d} - Ax^3$, which implies $0 = adx - bcx - dAx^3$, i.e., $x(ad - bc - dAx^2) = 0$. This has roots at $x = 0$ (the fixed point at the origin that we already knew about) and at $\pm x_p$ where $x_p = \sqrt{\frac{ad - bc}{dA}}$, provided thatthe expression under the radical is positive. That is, if the determinant $\det J = ad - bc$ is positive, there are two fixed points other than the origin, at $\pm(x_p, cx_p / d)$.

D. Linearize the system near those fixed points.

We need to approximate the right hand sides of

$$\begin{aligned} \frac{dx}{dt} &= ax - by - Ax^3 \\ \frac{dy}{dt} &= -cx + dy \end{aligned}$$

near $(x, y) = (x_p, y_p) = (x_p, cx_p / d)$. (The analysis at $(x, y) = -(x_p, cx_p / d)$ For $\frac{dx}{dt}$, we do this by Taylor series expansion:

$$\begin{aligned} ax - by - Ax^3 &\approx (x - x_p)\frac{\partial}{\partial x}(ax - by - Ax^3)\Big|_{(x,y)=(x_p,y_p)} + (y - y_p)\frac{\partial}{\partial y}(ax - by - Ax^3)\Big|_{(x,y)=(x_p,y_p)} \\ &= (x - x_p)(a - 3Ax^2)\Big|_{(x,y)=(x_p,y_p)} + (y - y_p)(-b)\Big|_{(x,y)=(x_p,y_p)} \\ &= (x - x_p)(a - 3Ax_p^2) - b(y - y_p) \\ &= (a - 3\frac{ad - bc}{d})(x - x_p) - b(y - y_p) \end{aligned}.$$

For $\frac{dy}{dt}$, this is a linear function of x and y, so the Taylor expansion is trivial and exact:

$\frac{dy}{dt} = -cx + dy = -c(x - x_p) + d(y - y_p)$, since $-cx_p + dy_p = 0$.

So, near $(x_p, y_p) = (x_p, cx_p / d)$, the system can be approximated by

$$\begin{aligned} \frac{dx}{dt} &= (a - 3\frac{ad - bc}{d})(x - x_p) - b(y - y_p) \\ \frac{dy}{dt} &= -c(x - x_p) + d(y - y_p) \end{aligned}.$$

E. What kind of behavior does the system have near those fixed points?

We need to find the eigenvalues of the matrix

$$J_p = \begin{pmatrix} a - 3\dfrac{ad-bc}{d} & -b \\ -c & d \end{pmatrix}$$, i.e., the roots of $\lambda^2 - \lambda\,\mathrm{trace}(J_p) + \det(J_p) = 0$

It has trace $\mathrm{trace}(J_p) = a + d - 3\dfrac{ad-bc}{d}$, and determinant

$$\det(J_p) = \left(a - 3\frac{ad-bc}{d} \right) d - bc = ad - 3(ad-bc) - bc = -2\det J .$$

$\lambda = \dfrac{\mathrm{trace}(J_p) \pm \sqrt{\mathrm{trace}(J_p)^2 - 4\det(J_p)}}{2}$. Since $\det J > 0$, $\det(J_p) < 0$, and the expression under the radical is necessarily positive, and larger than $\mathrm{trace}(J_p)$. Therefore the roots are both real, and one is positive and one is negative. Therefore the fixed point (x_p, y_p) is a saddle point. The same argument holds for the other fixed point, $(-x_p, -y_p)$.