Exam, 2016-2017

Do any 12 points (27 points total)

Q1: 3 parts, one point each
Q2: 4 parts, one point each
Q3: 5 parts, one point each
Q4: 5 parts, one point each
Q5: 1 part, worth 5 points
Q6: 5 parts, one point each

Question 1. The Frobenius norm

The Frobenius norm of a matrix M, $\|M\|_F$ which is an overall measure of the size of the matrix, is defined as the square root of the sum of the squares of the absolute values of its entries, i.e.,

$$\|M\|_F = \sqrt{\sum_{i,j} |m_{i,j}|^2} \ .$$

A. Show that $\|M\|_F = \sqrt{tr(MM^*)}$, where M^* is the adjoint of M.

B. If A is self-adjoint, show that $\|A\|_F = \sqrt{\sum \lambda_i^2}$, where the λ_i are the eigenvalues of A.

C. Show that if U is unitary, then $\|M\|_F = \|UMU^{-1}\|_F$, i.e., the Frobenius norm is invariant under unitary transformation.

Question 2. Representations one group inside of another group

Say G is a (finite) group, U is a representation (with $U: g \to U_g$ a homomorphism from G to unitary transformations in $Hom(V,V)$, and H is a proper subgroup G (i.e., H is a subgroup and $H \neq G$).

A. Show that U is a representation of H.

B. Show that if U is an irreducible representation of H, then it is also an irreducible representation of G.

C. If U is an irreducible representation of G, then is it necessarily an irreducible representation of H? If yes, provide a proof; if no, provide an example in which U becomes reducible when restricted to a subgroup.

D. Given the above setup, with U an irreducible representation of G, and further assume that the character of U is zero for all elements of G that are not in H (this behavior is not unusual, for example see part E of Q3 below). Then show that U is always reducible on H

Question 3. Another way to construct group representations

Here we will develop another way to construct group representations. Let G be a (finite) group, H is a subgroup, and recall that, for any $b \in G$ a "right coset" of H, denoted Hb, is the set of all elements of $g \in G$ that can be written as $g = hb$. We showed in class that G is the disjoint union of all of its distinct right cosets (including the right coset corresponding to the identity $H = He$). This means that the number of right cosets is $|G|/|H|$. Any element $c \in G$ can be viewed as acting as a permutation on the cosets, since $(Hb)c$ is also a coset, namely, $(Hb)c = H(bc)$.

A. Show that the action on cosets yields a permutation representation of G, that we will denote here as Q.

B. What does the above construction reduce to if $H = G$? What does it reduce to if H is just the identity element?

C. Assume that H is not all of G, i.e., that there is at least some element of G that is not in H. Show that the representation we constructed is NOT irreducible.

D. Now assume that H is a normal subgroup of G. (A "normal" subgroup is a subgroup for which every right coset is also a left coset, i.e., that $bH = Hb$. Equivalently, for a normal subgroup H and any $h \in H$ and any $g \in G$, then $ghg^{-1} \in H$.) Determine the character of $\chi_Q(h)$ for $h \in H$.

E. As in D, but now determine the character $\chi_Q(c)$ for $c \notin H$.

Question 4. Coherence and network identification

Say $S_1(t)$ and $S_2(t)$ are independent noise sources with power spectra $P_{S_1}(\omega)$ and $P_{S_2}(\omega)$, which are connected to two observable outputs $R_1(t)$ and $R_2(t)$ by the following network, where L_{ij} are linear filters with transfer functions $\tilde{L}_{ij}(\omega)$.
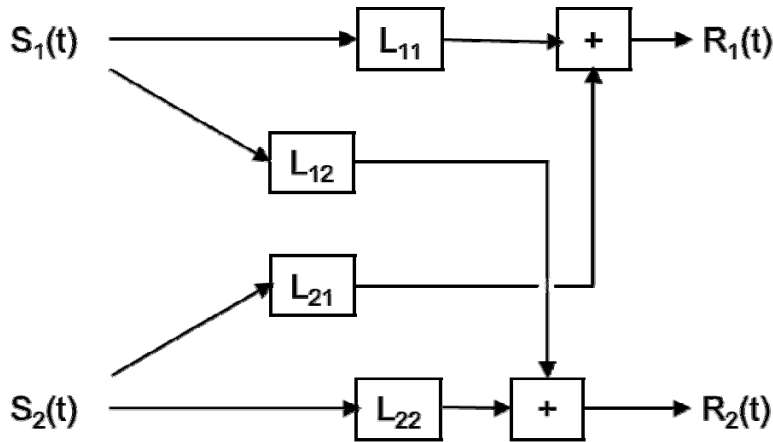
A. Find the power spectra $P_{R_1}(\omega)$ and $P_{R_2}(\omega)$ of the two outputs.

B. Find the cross-spectrum of $R_1(t)$ and $R_2(t)$.

C. Find the coherence of $R_1(t)$ and $R_2(t)$.

D. When is the magnitude of the coherence equal to 1?

E. Provide an interpretation for the answer in D.

Question 5. Coupled neurons (or populations): linear systems view

Consider the network below, in which one neural population (the "+" at the top) is self-exciting with linear dynamics specified by the impulse response $A(t)$, a second neural population (the "+" at the bottom) is self-exciting with linear dynamics specified by $D(t)$, and the two neurons are coupled to each other via the linear dynamics specified by the impulse responses $C(t)$ and $B(t)$.

Determine the transfer function between the input $s(t)$ and the output $x(t)$, in terms of the transfer functions $\tilde{A}(\omega)$, $\tilde{B}(\omega)$, $\tilde{C}(\omega)$, and $\tilde{D}(\omega)$.
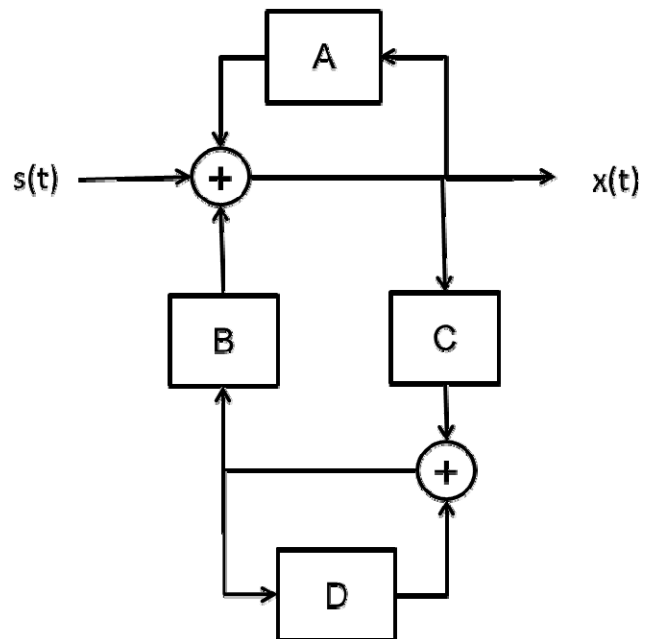
Question 6. Coupled neurons (or neural populations), dynamical systems view

Let $x(t)$ and $y(t)$ represent the fluctuations in activity of two neural populations; for convenience, we set the mean level of both to zero. Consider the following dynamics:

$$\frac{dx}{dt} = ax - by - Ax^3$$

, where $a > 0$, $d > 0$ (the populations are self-exciting), $b > 0$, $c > 0$ (so the

$$\frac{dy}{dt} = -cx + dy$$

populations inhibit each other), and $A > 0$ (so there is no runaway activity in either direction of $x(t)$).

A. What are the possible kinds of behavior near $(x, y) = (0,0)$, and for which parameter values do they occur?

B. Sketch the nullclines (the loci in the (x, y) plane in which $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$) for the regimes identified in A.

C. Are there any other fixed points for the regimes identified in A? Under what circumstances?

D. Linearize the system near those fixed points.

E. What kind of behavior does the system have near those fixed points?