Groups, Fields, and Vector Spaces

Homework #3 (2018-2019), Questions

Q1: Tensor products: concrete examples – preliminary for determinant

Let V and W be two-dimensional vector spaces, with bases $\{v_1, v_2\}$ and $\{w_1, w_2\}$. So $\{v_i \otimes w_j\}$ is a basis for $V \otimes W$. Say $x_i \in V$ has the basis expansion $x = \alpha_1 v_1 + \alpha_2 v_2$ and $y_i \in W$ has the basis expansion $y = \beta_1 w_1 + \beta_2 w_2$.

A. Expand $x \otimes y$ in the basis $\{v_i \otimes w_j\}$.

B. Now say $V = W$, and we are using the same basis for x and y, so that $x = \alpha_1 v_1 + \alpha_2 v_2$ and $y = \beta_1 v_1 + \beta_2 v_2$. Expand $x \otimes y$ in the basis $\{v_i \otimes v_j\}$.

C. Expand $x \otimes y + y \otimes x$ in the basis $\{v_i \otimes v_j\}$.

D. Expand $x \otimes y - y \otimes x$ in the basis $\{v_i \otimes v_j\}$.

Q2. Explicit construction of a 2 x 2 determinant

A similar setup of part B of Q1: $\{v_1, v_2\}$ is a basis for a two-dimensional space V. In this basis, the linear transformation M is defined by the matrix $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$, where $Mv_1 = m_{11} v_1 + m_{12} v_2$ and $Mv_2 = m_{21} v_1 + m_{22} v_2$. We are guaranteed that $anti(V^{\otimes 2})$ is one-dimensional, and that it is spanned by $\varphi = (v_1 \otimes v_2) - (v_2 \otimes v_1)$, so that $M\varphi$ is a multiple of φ. Compute this multiple, i.e., the determinant, by computing $M\varphi$.

Q3. Another finite field example

Recall that $\mathbb{Z}_2$ is the field containing $\{0,1\}$, with addition and multiplication defined (mod 2). Consider the polynomial $x^4 + x + 1 = 0$. This has no solutions in $\mathbb{Z}_2$, so let's add a formal quantity ξ for which $\xi^4 + \xi + 1 = 0$ (and which satisfies the associative, commutative, and distributive laws for addition and multiplication with itself and with $\{0,1\}$), and see whether it generates a field.

Using $\xi^4 + \xi + 1 = 0$, express ξ^r in terms of 1, ξ, ξ^2, and ξ^3 for $r = 1,...,15$.