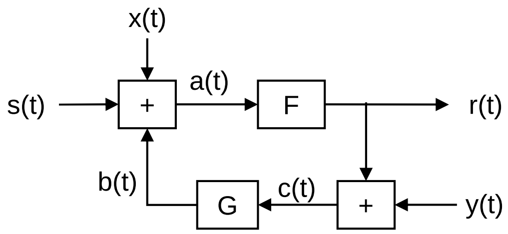
Linear Systems, Black Boxes, and Beyond

Homework #2 (2018-2019), Answers

Q1: Noises and networks

Given the following network, where F *and* G *are linear filters with transfer functions* $\tilde{F}(\omega)$ *and* $\tilde{G}(\omega)$, *and* $s(t)$, $x(t)$ *and* $y(t)$ *are independent noise inputs with power spectra* $P_S(\omega)$, $P_X(\omega)$, *and* $P_Y(\omega)$, *calculate the power spectrum* $P_R(\omega)$ *of* $r(t)$.



We want to determine $P_R(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle\left|\tilde{r}(\omega)\right|^2\right\rangle=\lim_{T\to\infty}\frac{1}{T}\left\langle\tilde{r}(\omega)\overline{\tilde{r}(\omega)}\right\rangle$, where $\tilde{r}(\omega)$ is a Fourier estimate of $r(t)$ over some finite but long interval *T*, i.e., $\tilde{r}(\omega)=\int_0^T e^{-i\omega t}r(t)dt$. To find $\tilde{r}(\omega)$ in terms of $\tilde{s}(\omega)$, $\tilde{x}(\omega)$ and $\tilde{y}(\omega)$, we chase Fourier estimates through the network, considering *T* to be long enough so that the Fourier estimates can be replaced by the corresponding Fourier transforms:

$$\tilde{a}(\omega)=\tilde{s}(\omega)+\tilde{x}(\omega)+\tilde{b}(\omega)$$
$$\tilde{r}(\omega)=\tilde{F}(\omega)\tilde{a}(\omega)=\tilde{F}(\omega)\left(\tilde{s}(\omega)+\tilde{x}(\omega)+\tilde{b}(\omega)\right)$$
$$\tilde{c}(\omega)=\tilde{r}(\omega)+\tilde{y}(\omega)=\tilde{F}(\omega)\left(\tilde{s}(\omega)+\tilde{x}(\omega)+\tilde{b}(\omega)\right)+\tilde{y}(\omega)$$
$$\tilde{b}(\omega)=\tilde{c}(\omega)\tilde{G}(\omega)=\tilde{F}(\omega)\tilde{G}(\omega)\left(\tilde{s}(\omega)+\tilde{x}(\omega)+\tilde{b}(\omega)\right)+\tilde{y}(\omega)\tilde{G}(\omega)$$

Solving the final equation for $\tilde{b}(\omega)$:

$\tilde{b}(\omega)\left(1-\tilde{F}(\omega)\tilde{G}(\omega)\right)=\tilde{F}(\omega)\tilde{G}(\omega)\left(\tilde{s}(\omega)+\tilde{x}(\omega)\right)+\tilde{y}(\omega)\tilde{G}(\omega)$, so

$$\tilde{b}(\omega)=\tilde{G}(\omega)\frac{\tilde{F}(\omega)\left(\tilde{s}(\omega)+\tilde{x}(\omega)\right)+\tilde{y}(\omega)}{1-\tilde{F}(\omega)\tilde{G}(\omega)}.$$

Using the above equation for $\tilde{r}(\omega)$:

$$\tilde{r}(\omega)=\tilde{F}(\omega)\left(\tilde{s}(\omega)+\tilde{x}(\omega)+\tilde{b}(\omega)\right)=\tilde{F}(\omega)\left(\tilde{s}(\omega)+\tilde{x}(\omega)+\tilde{G}(\omega)\frac{\tilde{F}(\omega)\left(\tilde{s}(\omega)+\tilde{x}(\omega)\right)+\tilde{y}(\omega)}{1-\tilde{F}(\omega)\tilde{G}(\omega)}\right)$$

$$=\tilde{F}(\omega)\left(\frac{\tilde{s}(\omega)+\tilde{x}(\omega)+\tilde{G}(\omega)\tilde{y}(\omega)}{1-\tilde{F}(\omega)\tilde{G}(\omega)}\right)=\frac{\tilde{F}(\omega)}{1-\tilde{F}(\omega)\tilde{G}(\omega)}\left(\tilde{s}(\omega)+\tilde{x}(\omega)+\tilde{G}(\omega)\tilde{y}(\omega)\right)$$

In computing $\left\langle\left|\tilde{r}(\omega)\right|^2\right\rangle=\left\langle\tilde{r}(\omega)\overline{\tilde{r}(\omega)}\right\rangle$, we will get terms involving just s just x, just y and cross-terms between them. The expected value of the cross-terms vanish because s, x, and y are hypothesized to be independent. So

$$\frac{1}{T}\left\langle\left|\tilde{r}(\omega)\right|^2\right\rangle=\left\langle\left|\frac{\tilde{F}(\omega)}{1-\tilde{F}(\omega)\tilde{G}(\omega)}\left(\tilde{s}(\omega)+\tilde{x}(\omega)+\tilde{G}(\omega)\tilde{y}(\omega)\right)\right|^2\right\rangle$$

$$=\frac{1}{T}\left|\frac{\tilde{F}(\omega)}{1-\tilde{F}(\omega)\tilde{G}(\omega)}\right|^2\left\langle\left|\left(\tilde{s}(\omega)+\tilde{x}(\omega)+\tilde{G}(\omega)\tilde{y}(\omega)\right)\right|^2\right\rangle$$

$$=\frac{1}{T}\left|\frac{\tilde{F}(\omega)}{1-\tilde{F}(\omega)\tilde{G}(\omega)}\right|^2\left\langle\left(\tilde{s}(\omega)+\tilde{x}(\omega)+\tilde{G}(\omega)\tilde{y}(\omega)\right)\overline{\left(\tilde{s}(\omega)+\tilde{x}(\omega)+\tilde{G}(\omega)\tilde{y}(\omega)\right)}\right\rangle \quad .$$

$$=\left|\frac{\tilde{F}(\omega)}{1-\tilde{F}(\omega)\tilde{G}(\omega)}\right|^2\left\langle\frac{1}{T}\left(\tilde{s}(\omega)\overline{\tilde{s}(\omega)}+\tilde{x}(\omega)\overline{\tilde{x}(\omega)}+\tilde{G}(\omega)\tilde{y}(\omega)\overline{\tilde{G}(\omega)\tilde{y}(\omega)}+cross-terms\right)\right\rangle$$

$$=\left|\frac{\tilde{F}(\omega)}{1-\tilde{F}(\omega)\tilde{G}(\omega)}\right|^2\left\langle\frac{1}{T}\left(\tilde{s}(\omega)\overline{\tilde{s}(\omega)}+\tilde{x}(\omega)\overline{\tilde{x}(\omega)}+\left|\tilde{G}(\omega)\right|^2\tilde{y}(\omega)\overline{\tilde{y}(\omega)}\right)\right\rangle$$

Taking the limit as $T\to\infty$,

$$P_R(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle\left|\tilde{r}(\omega)\right|^2\right\rangle$$

$$=\lim_{T\to\infty}\left|\frac{\tilde{F}(\omega)}{1-\tilde{F}(\omega)\tilde{G}(\omega)}\right|^2\left\langle\frac{1}{T}\left(\tilde{s}(\omega)\overline{\tilde{s}(\omega)}+\tilde{x}(\omega)\overline{\tilde{x}(\omega)}+\left|\tilde{G}(\omega)\right|^2\tilde{y}(\omega)\overline{\tilde{y}(\omega)}\right)\right\rangle.$$

$$=\left|\frac{\tilde{F}(\omega)}{1-\tilde{F}(\omega)\tilde{G}(\omega)}\right|^2\left(P_S(\omega)+P_X(\omega)+\left|\tilde{G}(\omega)\right|^2P_Y(\omega)\right)$$

Q2: Distinguishing signals

As mentioned in the notes, the power spectrum of a Poisson impulse train is flat (page 24 of LSBB). So is the power spectrum of white noise. If a Poisson impulse train and a white noise signal are filtered by the same linear filter, will the resulting power spectra be the same?

Yes, because the relationship between the input power spectrum and the output power spectrum does not depend on Gaussian-ness, as it follows merely from the frequency-domain view of how progressively longer samples of signals are transformed.

What this means is that signals with the same power spectrum may in fact be quite different. As in this example, they may have very distributions of the signal values. Deviations from Gaussian-ness can also be detected via the bispectrum and other extensions of the power spectrum, as mentioned in the notes (page 20 of LSBB).