Linear Transformations and Group Representations

Homework #1 (2018-2019), Answers

Q1: Eigenvectors and eigenvalues of time-translation

A. Consider two vectors c_ω and s_ω defined by $c_\omega(t)=\cos(\omega t)$ and $s_\omega(t)=\sin(\omega t)$, and the vector space V_ω that they span. As before, define $(D_T v)(t)=v(t+T)$. Show that $D_T c_\omega$ and $D_T s_\omega$ are in V_ω by displaying $D_T c_\omega$ and $D_T s_\omega$ as linear combinations of c_ω and s_ω.

$(D_T c_\omega)(t)=c_\omega(t+T)=\cos(\omega(t+T))=\cos\omega T\cos\omega t-\sin\omega T\sin\omega t=(\cos\omega T)c_\omega(t)-(\sin\omega T)s_\omega(t)$
so
$D_T c_\omega=(\cos\omega T)c_\omega-(\sin\omega T)s_\omega$.

Similarly,
$(D_T s_\omega)(t)=s_\omega(t+T)=\sin(\omega(t+T))=\sin\omega T\cos\omega t+\cos\omega T\sin\omega t=(\sin\omega T)c_\omega(t)+(\cos\omega T)s_\omega(t)$,
so
$D_T s_\omega=(\sin\omega T)c_\omega+(\cos\omega T)s_\omega$.

B. Express D_T as a 2×2 matrix, using c_ω and s_ω as a basis.
From Part A,
$$\begin{array}{l} D_T c_\omega=(\cos\omega T)c_\omega-(\sin\omega T)s_\omega \\ D_T s_\omega=(\sin\omega T)c_\omega+(\cos\omega T)s_\omega \end{array}, \text{ so}$$
$$D_T\begin{pmatrix} c_\omega \\ s_\omega \end{pmatrix}=\begin{pmatrix} \cos\omega T & -\sin\omega T \\ \sin\omega T & \cos\omega T \end{pmatrix}\begin{pmatrix} c_\omega \\ s_\omega \end{pmatrix}.$$

C. Write the characteristic equation for the 2×2 matrix in part B.
The characteristic equation for $D_T=\begin{pmatrix} \cos\omega T & -\sin\omega T \\ \sin\omega T & \cos\omega T \end{pmatrix}$ is $\det(zI-D_T)=0$.

$$\det(zI-D_T)=\det\begin{pmatrix} z-\cos\omega T & \sin\omega T \\ -\sin\omega T & z-\cos\omega T \end{pmatrix}=(z-\cos\omega T)^2-(\sin\omega T)(-\sin\omega T)$$
$$=z^2-2\cos\omega T+(\cos\omega T)^2+(\sin\omega T)^2$$
$$=z^2-2\cos\omega T+1$$
, so the characteristic equation is $z^2-2\cos\omega T+1=0$.

D. Solve the characteristic equation in Part C to determine the eigenvalues of D_T in V_ω.
Using the quadratic formula, $z^2-2\cos\omega T+1=0$ solves for

$z=\frac{2\cos\omega T\pm\sqrt{4\cos^2\omega T-4}}{2}=\cos\omega T\pm\sqrt{\cos^2\omega T-1}=\cos\omega T\pm i\sqrt{1-\cos^2\omega T}$, where the last step is justified because $\cos^2\omega T-1\le 0$. So

$z=\cos\omega T\pm i\sqrt{1-\cos^2\omega T}=\cos\omega T\pm i\sin\omega T=e^{\pm i\omega T}$.

E. Show that $c_\omega \pm is_\omega$ are eigenvectors of D_T.

Using Part A,

$D_T\left(c_\omega+is_\omega\right)=D_T\left(c_\omega\right)+iD_T\left(s_\omega\right)=\left((\cos\omega T)c_\omega-(\sin\omega T)s_\omega\right)+i\left((\sin\omega T)c_\omega+(\cos\omega T)s_\omega\right)$.

Collecting terms,

$\left((\cos\omega T)c_\omega-(\sin\omega T)s_\omega\right)+i\left((\sin\omega T)c_\omega+(\cos\omega T)s_\omega\right)=$

$\left(\cos\omega T+i\sin\omega T\right)c_\omega+\left(-\sin\omega T+i\cos\omega T\right)s_\omega=$

$\left(\cos\omega T+i\sin\omega T\right)c_\omega+\left(i\sin\omega T+\cos\omega T\right)is_\omega=$.

$\left(\cos\omega T+i\sin\omega T\right)(c_\omega+is_\omega)=e^{i\omega T}(c_\omega+is_\omega)$

So $D_T(c_\omega+is_\omega)=e^{i\omega T}(c_\omega+is_\omega)$, as required.

Similarly, $D_T(c_\omega-is_\omega)=e^{-i\omega T}(c_\omega-is_\omega)$. We didn't have to check this, since the assignment of i vs. $-i$ is arbitrary (i.e., complex-conjugation is an automorphism), and this switch leaves D_T invariant.

Q2: Eigenvectors and eigenvalues of the derivative

A. Setup is the same as Q1, but with the transformation Bv defined by $\left(Bv\right)(t)=v'(t)$, i.e., the derivative, rather than D_T. Display Bc_ω and Bs_ω as linear combinations of c_ω and s_ω.

$\left(Bc_\omega\right)(t)=\frac{d}{dt}(\cos\omega t)=-\omega\sin\omega t=-\omega s_\omega(t)$.

$\left(Bs_\omega\right)(t)=\frac{d}{dt}(\sin\omega t)=\cos\omega t=\omega c_\omega(t)$.

B. Express B as a 2×2 matrix, using c_ω and s_ω as a basis.

From Part A,

$B\begin{pmatrix}c_\omega\\ s_\omega\end{pmatrix}=\begin{pmatrix}0 & -\omega\\ \omega & 0\end{pmatrix}\begin{pmatrix}c_\omega\\ s_\omega\end{pmatrix}$.

C. Write the characteristic equation for the 2×2 matrix in part B.

The characteristic equation for $B=\begin{pmatrix}0 & -\omega\\ \omega & 0\end{pmatrix}$ is $\det\left(zI-B\right)=0$.

$\det\left(zI-B\right)=\det\begin{pmatrix}z & \omega\\ -\omega & z\end{pmatrix}=z^2+\omega^2$, so the characteristic equation is $z^2+\omega^2=0$.

D. Solve the characteristic equation in Part C to determine the eigenvalues of B in V_ω.

$z^2+\omega^2=0$ solves for $z=\pm i\omega$.

E. Show that $c_\omega \pm is_\omega$ *are eigenvectors of* B.

Using Part A, $B(c_\omega + is_\omega) = B(c_\omega) + iB(s_\omega) = (-\omega s_\omega) + i(\omega c_\omega) = i\omega(c_\omega + is_\omega)$.

So $B(c_\omega + is_\omega) = i\omega(c_\omega + is_\omega)$, as required. Similarly, $B(c_\omega - is_\omega) = -i\omega(c_\omega - is_\omega)$.