Linear Transformations and Group Representations

Homework #2 (2018-2019), Questions

Q1: Some representations of the "continuous dihedral" group.

Let G be the "continuous dihedral" group , i.e., the group of rotations and reflections of a circle. For definiteness, let R_θ be a clockwise rotation by θ, and let M be the reflection in the vertical axis (that sends x to $-x$ and preserves y). The group consists of R_θ, M, and all the transformations that can be generated by composing them.

A. Verify geometrically that these group elements satisfy $R_\theta R_\phi = R_{\theta+\phi}$, $R_\theta M = MR_{-\theta}$, and $M^2 = I$ (the identity).

B. Show that any element of the group is equal either to R_ϕ or $R_\phi M$, for some ϕ.

C. Geometrically, what is the transformation $R_\theta M R_\theta^{-1}$?What is its reduction to the form specified in part B?

D. Write R_θ and M as 2×2 matrices, and thereby construct a 2-dimensional unitary representation L of G. Verify the identities of part A algebraically.

E. What is the character of R_θ, $R_\theta M$, and $R_\theta M R_\theta^{-1}$ in the representation L?

F. Define $L_{R_\theta}^{[n]} = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix} = R^n = R_{n\theta}$ and $L_M^{[n]} = M$ (the latter is independent of n). Show that $L^{[n]}$is a representation. Note that to do this, it suffices to show that the mapping from group elements to the unitary matrices defined by $L^{[n]}$ will preserve the rules that govern group operations: $R_\theta R_\phi = R_{\theta+\phi}$, $R_\theta M = MR_{-\theta}$, and $M^2 = I$.

G. Define $S_{R_\theta} = 1$ and $S_M = -1$. Show S is a one-dimensional representation.

Q2: Characters of representations of a permutation group.

Let P be the permutation group on three objects. This has six elements.

A. Write each group element as a 3×3 permutation matrix. As discussed, this is a unitary representation, which we can call U. For each of the six permutations σ, determine the character $\chi_U(\sigma)$.

B. Consider the subgroup of G generated by R_θ and M, where θ is restricted to 0, $2\pi/3$, and $4\pi/3$. Show that this is the permutation group on 3 objects.

C. Restricting the group representation L of Question 1 (parts D and E) to the subgroup in part B yields a 2-dimensional unitary representation of P. Determine its character for the six group elements of P.

D. Restricting the group representation S of Question 1 (part G) to the subgroup in part B yields a 1-dimensional unitary representation of P. Determine its character for the six group elements of P.